The Canonical Triatrad of the Natural Numbers

A four-tree decomposition and the open problem of terminal class assignment

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Abstract

This paper investigates a deterministic recursive descent on the positive natural numbers. For each current value N , a uniquely determined integer index

$$t(N) = \left\lfloor \frac{\sqrt{4N+1}+1}{2} \right\rfloor$$

is assigned. Subsequently, the triangular number

$$T_{t(N)} = \frac{t(N)(t(N)+1)}{2}$$

of$N$ is subtracted. The resulting descent operator is

$$F(N) = N - T_{t(N)}.$$

The operator is applied recursively as long as its result remains positive. It is shown that every positive natural starting value reaches exactly one of the four terminal values after a finite number of steps

$$\mathcal{T} = \{1,2,3,6\}$$

. This complete set of four terminal values is referred to as a canonical triatrad. The corresponding terminal value$E(N)$ uniquely partitions the positive natural numbers into four disjoint classes

$$\mathbb{N}_{>0} = \mathcal{C}_1 \dot{\cup} \mathcal{C}_2 \dot{\cup} \mathcal{C}_3 \dot{\cup} \mathcal{C}_6,$$

where

$$\mathcal{C}_e = \{N \in \mathbb{N}_{>0} : E(N) = e\}$$

the following holds for every $e \in \mathcal{T}$. Since every non-terminal value has exactly one successor, whilst a value may have several predecessors, the four classes form four directed backward trees with roots 1 ,2 ,3 and 6 .

In addition to the formal derivation of the operator, the proof of its finiteness and the demonstration of the completeness of the terminal set, this paper investigates the empirical structure of the four classes. To this end, extensive numerical test series extending into large number ranges are analysed. The focus is on the global class weights, their local variation, the occurrence of long class-free intervals, and the question of whether the four trees possess asymptotically stable or structurally varying distributions.

Whilst the terminal set and the four-tree decomposition follow from the operator, the direct class assignment of any given initial value remains open. To date, the terminal value can only be determined by executing the descent. The central follow-up question is therefore which intrinsic arithmetic property of a natural number N determines whether it belongs to the class $\mathcal{C}_1$,$\mathcal{C}_2$,$\mathcal{C}_3$ or $\mathcal{C}_6$. The mathematical core thus consists of an elementary operator, a universal terminal set of four, a complete four-tree decomposition and an open direct classification problem .

Keywords

Triatrad; natural numbers; triangular numbers; recursive descent; terminal values; value function; four-tree decomposition; discrete dynamics; class distribution; voids; direct class assignment; Arithmetic Dynamics; Collatz-like systems; bit-level reduction; basins of attraction; partition of natural numbers

Foreword

The canonical triatrad was not discovered through a deliberate search for a partition of the natural numbers into four classes. Its origin lies in the investigation of descent movements within the previously developed KQ and KFS structures. In this context, the terminal set

$$\{1,2,3,6\}$$

repeatedly emerged. It was only the subsequent algebraic simplification of the descent used that showed that the underlying process can be formulated, independently of KQ and KFS, as elementary recursive subtraction of uniquely determined triangular numbers.

This distinction is essential to the present work. The history of the discovery of the triatrad is linked to KQ and KFS; however, the main mathematical subject matter does not require these systems. The operator, its finiteness, the complete terminal set and the resulting four-tree decomposition can be described exclusively using classical integer functions, triangular numbers and elementary inequalities. The KQ and KFS perspectives are therefore only addressed after the independent mathematical presentation of the triatrad.

The term ‘triatrad’ links the operator’s origin in triangular numbers with the terminal set consisting of four elements. Formally, the triatrad denotes exclusively the set

$$\mathcal{T} := \{1,2,3,6\}.$$

Furthermore, however, the subject of this research encompasses the entire structure generated by the operator, comprising the terminal function, four classes and four directed trees. The triatrad is therefore not merely a notable set of four, but the common terminal root system of a recursive dynamical system on the positive natural numbers. This paper consistently distinguishes between proven statements, numerical findings and open structural questions. The finiteness of the descent, the completeness of the terminal set and the resulting unique four-tree decomposition are proven. In particular, the frequencies and local distributions of the four classes are investigated empirically. It remains an open question why a specific value N is assigned precisely to a particular Triatrad terminal value, and whether this assignment can be described by a direct arithmetic rule.

Familiarity with the underlying KQ-KFS paper is assumed for the reading of this text: (Meißner, 2025b) .

It should also be noted that the concept of mono-information already describes an initial concrete application of the KQ and KFS principles and explores their information-theoretical potential (Meißner, 2025c) .

The paper ‘Geometries of the Factorisation Problem: An Analytical Approach via KQ Groups – Version 3’ is also part of an ongoing analytical study (Meißner, 2025a) .

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1. Introduction

1.1 Subject of the thesis

The set of positive natural numbers can be decomposed into four disjoint terminal classes by means of a simple recursive operation. The starting point is an integer function$t(N)$, which assigns an index to every positive natural number$N$. This index determines a triangular number, which is subtracted from$N$ in the next step. The operation is repeated with the newly generated positive value.

Although each individual step consists solely of determining an integer index and subtracting a triangular number, the complete process possesses a striking global property: after a finite number of steps, every positive natural starting value reaches exactly one of the four values

$$1, \qquad 2, \qquad 3, \qquad 6.$$

No other positive final values are possible. Together, the four values form the canonical triatrad

$$\mathcal{T} = \{1,2,3,6\}.$$

The final value reached in each instance defines a function

$$E: \mathbb{N}_{>0} \to \mathcal{T}.$$

Thus, every positive natural value is uniquely assigned to one of the four classes

$$\mathcal{C}_1, \qquad \mathcal{C}_2, \qquad \mathcal{C}_3, \qquad \mathcal{C}_6$$

. Since the descent is deterministic, every non-terminal value has exactly one successor. Conversely, several larger values may descend to the same successor. The four classes therefore not only take the form of ordinary sets, but also form four branched, directed backward trees with roots 1 ,2 ,3 and 6 . The resulting structure is referred to in this paper as the four-tree decomposition of the positive natural numbers.

1.2 Objectives

The first objective of this paper is to provide a complete formal definition of the recursive triangular descent. Subsequently, its finiteness and the uniqueness of the

terminal value associated with each initial value are demonstrated. Building on this, it is proved that the set$\{1,2,3,6\}$ contains all possible positive terminal values.

The second objective is to describe the four-tree decomposition arising from the terminal value function. To this end, the four classes are defined, their disjunction and completeness are demonstrated, and the conditions under which a value can occur as a predecessor of an already given tree node are investigated.

The third objective is the empirical investigation of the four classes across large number ranges. The numerical data are not intended merely to confirm that all computed descents terminate in a value of the triatrad. Rather, the study examines the relative weights of the four classes, their behaviour as$N$ increases, local changes in the distribution, and extended intervals in which a particular class is completely absent. Such class-free intervals are referred to as triatrad voids.

The fourth objective is to formulate precisely the as yet unsolved direct classification problem. The value$E(N)$ can be calculated unambiguously through recursive application of the operator. However, no closed-form rule is yet known that determines the final value directly from$N$ or from any other simple arithmetic property. The question

Welche intrinsische Eigenschaft von N bestimmt $E(N)$?

therefore constitutes the central set of open problems in this paper.

1.3 Mathematical status of the statements

This paper distinguishes between three levels of knowledge. The first level comprises the mathematical statements arising from the definition of the operator. These include the uniqueness of every descent path, its finiteness, the complete terminal set and the Vierbaum decomposition.
The second level comprises numerical results. These include the measured frequencies of the four classes, their local distributions, the observed void structures and the descent time investigated using selected test values. These statements initially apply to the test ranges explicitly specified in each case.

The third level comprises open hypotheses and further structural questions. These include possible direct class invariants, asymptotic distribution rules, the question of unboundedly growing voids, and possible geometric interpretations of the four trees. Such considerations are not treated as consequences of the Triatrad theorem, but are explicitly identified as research questions.

1.4 Structure of the thesis

Chapters 2 to 7 develop the mathematical core independently of KQ and KFS. Following the arithmetic foundations, the descent operator is defined. Subsequently, finiteness, the terminal set, the limit function and the four-tree decomposition are derived.

Chapters 8 to 18 deal with the direct classification problem and the numerical investigations. The statistical section covers global and local class weights, in particular the conspicuous so-called 6-voids, amongst others, as well as supplementary observations on descent duration.

Chapter 19 describes the history of discovery within KQ and KFS and shows how the operator can be separated from these original representations and formulated in classical terms, as well as addressing further structural questions. The concluding chapters deal with further structural questions, open problems and the significance of the Triatrad as an object of discrete dynamics on the positive natural numbers.

2. Arithmetic Foundations

2.1 Triangular numbers

For every natural number$m \geq 1$, the$m$ th triangular number is defined by

$$T_m = 1 + 2 + \cdots + m = \frac{m(m+1)}{2}.$$

The first triangular numbers are

$$T_1 = 1, \qquad T_2 = 3, \qquad T_3 = 6, \qquad T_4 = 10, \qquad T_5 = 15.$$

In the recursive triangle number descent, the same triangle number is not always used. The index$m$ is recalculated at each step from the current value. This results in an adaptive operator whose subtraction value is linked to the initial value of the respective step.

2.2 The index $t(N)$

For each$N \in \mathbb{N}_{>0}$,

$$t(N) = \left\lfloor \frac{\sqrt{4N+1}+1}{2} \right\rfloor$$

defined.

The value$t(N)$ is the uniquely determined positive integer$m$ for which

$$m(m-1) \leq N < m(m+1)$$

holds. Equivalently, the interval, including its integer endpoints, can be written as

$$m(m-1) \leq N \leq m(m+1) - 1.$$

The positive natural numbers are thereby divided into consecutive intervals. For the first few values of$m$, we obtain

$$m = 1 \quad \Leftrightarrow \quad 1 \leq N \leq 1,$$

$$m = 2 \quad \Leftrightarrow \quad 2 \leq N \leq 5,$$

$$m = 3 \quad \Leftrightarrow \quad 6 \leq N \leq 11,$$

$$m = 4 \quad \Leftrightarrow \quad 12 \leq N \leq 19.$$

Within each of these intervals,$t(N)$ has a constant value. Consequently, the same triangular numberT_m is used for all values within the same interval.

2.3 Position within an interval

Let$m = t(N)$. Then every$N$ has a unique representation within the corresponding interval

$$N = m(m-1) + r$$

where

$$0 \leq r \leq 2m - 1.$$

The value$r$ denotes the position of$N$ within the interval defined by$m$. This representation allows for a linear form of the descent operator within each individual interval and is used in Chapter 3 for further derivation.

2.4 Uniqueness of the interval assignment

The intervals

$$I_m = \{N \in \mathbb{N}_{>0} : m(m-1) \leq N < m(m+1)\}$$

are pairwise disjoint and cover the positive natural numbers completely. The upper end ofI_m is

$$m(m+1) - 1,$$

whilst the subsequent intervalI_{m+1} begins at

$$(m+1)m = m(m+1)$$

. Consequently, there is neither a gap nor an overlap between any two consecutive intervals.

For every$N \in \mathbb{N}_{>0}$, there is thus exactly one$m \in \mathbb{N}_{>0}$ such that

$$N \in I_m.$$

This$m$ is equal to$t(N)$. The mapping

$$N \mapsto t(N)$$

is therefore unique.

3. The recursive descent of triangular numbers

3.1 Definition of the descent operator

Let$N \in N_{>\mathbb{0}}$ and

$$m = t(N).$$

The adaptive triangular number associated with$N$ is

$$T_{t(N)} = T_m = \frac{m(m+1)}{2}.$$

The descent operator is defined by

$$F(N) = N - T_{t(N)}.$$

Thus, the complete formula is

$$F(N) = N - \frac{t(N)(t(N)+1)}{2}.$$

At each step, the operator uses the triangular number whose index is determined by the current value itself. It thus differs from a recursion with a constant subtraction value, as well as from a standard greedy algorithm, in which the largest triangular number that does not exceed the current value is selected in each case.
For the Triatrad descent,$F(N)$ is only adopted as the next positive descent value if

$$F(N) > 0$$

applies. If, on the other hand,

$$F(N) \leq 0,$$

then the positive descent path ends at the current value N . A terminal value is therefore a positive natural number whose next formal operator value is no longer positive.

3.2 Recursive descent sequence

$N_0 \in \mathbb{N}_{>0}$A sequence is

$$N_0, N_1, N_2, \ldots$$

is generated recursively. As long as the next value is positive, the following applies

$$N_{k+1} = F(N_k).$$

The recursion ends at the first index$r$ for which

$$F(N_r) \leq 0$$

holds. The last positive valueN_r is referred to as the terminal value of the initial valueN_0 .

The terminal value function is defined by

$$E(N_0) = N_r$$

.

Thus,$E(N)$ is not the first non-positive value that the formally continued subtraction would produce. Rather, the terminal value is the last positive value immediately before leaving$\mathbb{N}_{>0}$.

3.3 Representation of the operator via the interval position

Let us again consider

$$m = t(N)$$

and

$$N = m(m-1) + r$$

with

$$0 \leq r \leq 2m - 1.$$

Then it follows that

$$\begin{aligned} F(N) &= m(m-1) + r - \frac{m(m+1)}{2} \\ &= \frac{2m(m-1) - m(m+1)}{2} + r \\ &= \frac{m(m-3)}{2} + r. \end{aligned}$$

Thus, within the intervalI_m , the operator takes the linear form

$$F(N) = \frac{m(m-3)}{2} + r.$$

The non-linear part of the entire descent thus lies exclusively in the re-determination of the interval index$m = t(N)$ after each step. Within a fixed interval,F acts as a simple shift by the constant triangular numberT_m .

3.4 Size reduction of a descent

For every positive value N , the following holds

$$T_{t(N)} > 0.$$

It therefore follows immediately that

$$F(N) < N.$$

As soon as$F(N)$ is positive and is adopted as the next descent value, the following therefore holds

$$0 < F(N) < N.$$

Every step actually taken strictly reduces the current value. The sequence of positive descent values is therefore strictly monotonically decreasing.

The magnitude of the subtracted triangular number is approximately half the current value. From

$$m(m-1) \leq N < m(m+1)$$

it follows that for large$m$, the order of magnitude is

$$N \approx m^2.$$

At the same time, the following holds

$$T_m = \frac{m(m+1)}{2} \approx \frac{m^2}{2}.$$

Thus, the next value is approximately of the form

$$F(N) \approx \frac{N}{2}.$$

This approximation explains why the observed number of descent steps in test values of different orders of magnitude is often close to the bit length of the initial value. However, it does not determine which of the four terminal values will be reached. The terminal class assignment depends on the discrete deviations that arise during the individual steps.

3.5 Example of a complete descent

For

$$N_0 = 589$$

,

$$t(589) = \left\lfloor \frac{\sqrt{2357} + 1}{2} \right\rfloor = 24.$$

The corresponding triangular number is

$$T_{24} = \frac{24 \cdot 25}{2} = 300.$$

It follows that

$$N_1 = 589 - 300 = 289.$$

The new value is

$$t(289) = 17$$

and therefore

$$T_{17} = \frac{17 \cdot 18}{2} = 153.$$

Thus, we obtain

$$N_2 = 289 - 153 = 136.$$

The complete sequence is

$$589 \to 289 \to 136 \to 58 \to 22 \to 7 \to 1.$$

The indices used for the six descent transitions carried out are

$$24,\ 17,\ 12,\ 8,\ 5,\ 3$$

For the terminal value1 , the following then holds:$t(1) = 1$ and$F(1) = 0$.

The reduced triangular numbers are

$$300,\ 153,\ 78,\ 36,\ 15,\ 6.$$

For the last positive value,1 , the following applies

$$t(1) = 1$$

and

$$F(1) = 1 - T_1 = 0.$$

The descent path therefore ends at

$$E(589) = 1.$$

3.6 Further brief examples

For$N = 4$, the following holds

$$t(4) = 2$$

and

$$F(4) = 4 - T_2 = 4 - 3 = 1.$$

Since

$$F(1) = 0,$$

it follows that

$$4 \to 1$$

and therefore

$$E(4) = 1.$$

For$N = 5$, we obtain

$$5 \to 2,$$

because

$$F(5) = 5 - 3 = 2$$

and

$$F(2) = 2 - 3 = -1.$$

Thus, the following holds

$$E(5) = 2.$$

For$N = 9$,

$$t(9) = 3$$

and

$$F(9) = 9 - T_3 = 9 - 6 = 3.$$

Since

$$F(3) = 3 - T_2 = 0,$$

it follows that

$$E(9) = 3.$$

For$N = 16$, the following holds

$$t(16) = 4$$

and

$$F(16) = 16 - T_4 = 16 - 10 = 6.$$

Since

$$F(6) = 6 - T_3 = 0,$$

it follows that

$$E(16) = 6.$$

These four examples alone illustrate the four possible terminal classes:

$$4 \in \mathcal{C}_1, \qquad 5 \in \mathcal{C}_2, \qquad 9 \in \mathcal{C}_3, \qquad 16 \in \mathcal{C}_6.$$

4. Finiteness and uniqueness of descent

4.1 Strict Monotonicity

LetN_k be a non-terminal value of a descent sequence. Then the following holds

$$F(N_k) > 0.$$

Because

$$T_{t(N_k)} > 0$$

it follows that

$$F(N_k) < N_k.$$

Thus, for every descent step actually carried out, the following holds

$$0 < N_{k+1} < N_k.$$

The positive terms of every descent sequence therefore form a strictly decreasing sequence of natural numbers.

4.2 Finiteness

An infinite strictly decreasing sequence of positive natural numbers does not exist. Consequently, the recursion cannot generate an infinite number of positive descent values.

For every initial value$N_0 \in \mathbb{N}_{>0}$, there therefore exists a finite index$r$ such that

$$F(N_r) \leq 0.$$

The positive descent path ends atN_r after a finite number of steps.

Thus, the limit function

$$E: \mathbb{N}_{>0} \to \mathbb{N}_{>0}$$

is well-defined for every positive natural starting value.

4.3 Exclusion of cycles

A cycle of length$s \geq 1$ would yield positive values

$$N_0, N_1, \dots, N_{s-1}$$

with

$$F(N_i) = N_{i+1}$$

and

$$F(N_{s-1}) = N_0$$

.

However, strict monotonicity would imply

$$N_0 > N_1 > \cdots > N_{s-1} > N_0.$$

This inequality is impossible. The operator therefore has no cycles within the positive natural numbers.

In particular, there are no non-terminal fixed points. From

$$F(N) = N$$

it would

$$T_{t(N)} = 0$$

follows, which is ruled out for $t(N) \geq 1$.

4.4 Uniqueness of the descent path

The function $t(N)$ assigns exactly one index to every positive natural number. This index determines exactly one triangular number. Consequently, every N has exactly one formal successor

$$F(N).$$

Since all subsequent values are again calculated unambiguously from their respective predecessors, the entire descent sequence for a given initial value is uniquely determined.

For each

$$N \in \mathbb{N}_{>0}$$

there is therefore exactly one terminal value

$$E(N).$$

Two different initial values may converge on the same intermediate value. From this common node onwards, their descent paths continue identically. A single initial value, on the other hand, can neither choose between several paths nor have several terminal values.

4.5 Theorem on Existence and Uniqueness

Theorem 1. For every positive natural number N , there exists, under the recursive triangular number descent, a uniquely determined finite descent path and exactly one terminal value $E(N)$.

Proof. The mapping $N \mapsto t(N)$ is unique. Therefore,

$$F(N) = N - T_{t(N)}$$

unique. By virtue of $T_{t(N)} > 0$, every positive successor is strictly less than its predecessor. The sequence of positive descent values is therefore strictly decreasing and must terminate after a finite number of steps. Since there is exactly one successor at each step, the last positive value is also unique. Thus, for every $N \in \mathbb{N}_{>0}$, there exists exactly one terminal value $E(N)$. ▫

5. The canonical triatrad

5.1 Terminal values

A value $N \in \mathbb{N}_{>0}$ is called terminal if its next formal operator value is not positive. It therefore holds that

$$N \text{ ist terminal} \quad \Leftrightarrow \quad F(N) \leq 0.$$

With

$$F(N) = N - T_{t(N)}$$

this is equivalent to

$$N \leq T_{t(N)}.$$

Let

$$m = t(N).$$

Then a terminal value must satisfy both conditions

$$m(m-1) \leq N < m(m+1)$$

and

$$N \leq \frac{m(m+1)}{2}$$

at the same time.

Consequently, a terminal value can only exist if the lower end of the intervalI_m does not lie above the corresponding triangular number.

5.2 Boundary for the terminal index

From

$$m(m-1) \leq N$$

and

$$N \leq \frac{m(m+1)}{2}$$

it follows that

$$m(m-1) \leq \frac{m(m+1)}{2}.$$

Since$m > 0$ holds, we can divide by$m$:

$$m - 1 \leq \frac{m + 1}{2}.$$

Multiplying by2 gives

$$2m - 2 \leq m + 1.$$

It therefore follows that

$$m \leq 3.$$

Every terminal value must therefore lie within one of the first three intervals

$$I_1, \qquad I_2, \qquad I_3$$

. For all

$$m \geq 4$$

, no terminal value is possible.

5.3 Terminal values for $m = 1$

For

$$m = 1$$

, the interval is

$$I_1 = \{1\}.$$

The corresponding triangular number is

$$T_1 = 1.$$

Therefore,

$$F(1) = 1 - 1 = 0.$$

The only terminal value for this level is

$$1.$$

5.4 Terminal values for $m = 2$

For

$$m = 2$$

the interval is

$$I_2 = \{2,3,4,5\}.$$

The corresponding triangular number is

$$T_2 = 3.$$

Terminal values are precisely those values$N \in I_2$ for which

$$N \leq 3$$

holds. This gives

$$N = 2$$

and

$$N = 3.$$

In fact,

$$F(2) = 2 - 3 = -1$$

and

$$F(3) = 3 - 3 = 0.$$

The terminal values of the second stage are therefore

$$2, \qquad 3.$$

5.5 Terminal values for $m = 3$

For

$$m = 3$$

the interval is

$$I_3 = \{6,7,8,9,10,11\}.$$

The corresponding triangular number is

$$T_3 = 6.$$

The terminal condition

$$N \leq 6$$

is satisfied within this interval exclusively by

$$N = 6$$

.

It holds that

$$F(6) = 6 - 6 = 0.$$

The only terminal value of the third stage is therefore

$$6.$$

5.6 Exclusion of further terminal values

For$m \geq 4$, the following holds

$$m(m-1) > \frac{m(m+1)}{2}.$$

Every value in the intervalI_m therefore satisfies

$$N \geq m(m-1) > T_m.$$

Thus,

$$F(N) = N - T_m > 0.$$

No value in an interval containing$m \geq 4$ can be terminal.
The complete set of positive terminal values is therefore

$$\{1,2,3,6\}.$$

5.7 Definition of the triatrad

The set

$$\mathcal{T} := \{1,2,3,6\}$$

is called the canonical triatrad of the recursive triangular number descent.
The term ‘canonical’ indicates that the set is not freely chosen. It follows uniquely and completely from the definition of the operator and the condition that the positive descent path ends at the last positive value.

5.8 Canonical Triatrad Theorem

Theorem 2. For every$N \in \mathbb{N}_{>0}$, the following holds

$$E(N) \in \{1,2,3,6\}.$$

The set

$$\mathcal{T} = \{1,2,3,6\}$$

is the complete set of all positive terminal values of the recursive triangular number descent.

Proof. By Theorem 1, every positive descent path ends at a terminal value after a finite number of steps. For a terminal value with$m = t(N)$, the following hold

$$m(m-1) \leq N$$

and

$$N \leq T_m = \frac{m(m+1)}{2}.$$

It follows that$m \leq 3$. The three possible stages yield the terminal sets

$$m = 1: \quad \{1\},$$

$$m = 2: \quad \{2,3\},$$

and

$$m = 3: \quad \{6\}.$$

For$m \geq 4$, no terminal value is possible. Thus, the complete terminal set is

$$\{1,2,3,6\}.$$

Consequently, for every $N \in \mathbb{N}_{>0}$

$$E(N) \in \mathcal{T}.$$

5.9 Significance of the theorem

The Triatrad Theorem fully answers the question of which positive terminal values are at all possible under the operator. There is no fifth terminal value.
However, the theorem does not answer which of the four terminal values corresponds to a specifically given initial value. The statements

$$E(N) \in \{1,2,3,6\}$$

and

$$E(N) = e$$

for a specific $e \in \mathcal{T}$ must be logically distinguished from one another. The first statement is clarified by the Triatrad theorem. The second, however, has so far required the actual execution of the recursive descent.

This immediately gives rise to the central open classification problem:

Welche arithmetische Eigenschaft von N bestimmt seinen Wert $E(N)$?

6. The terminal value function and the four triatrad classes

6.1 Definition of the terminal value function

According to Theorem 1, every positive natural number$N$ has exactly one terminal value. According to Theorem 2, this terminal value always belongs to the canonical triatrad

$$\mathcal{T} = \{1,2,3,6\}.$$

Thus, the terminal value function

$$E: \mathbb{N}_{>0} \to \mathcal{T}$$

is well-defined.

For the four terminal values, the following holds immediately

$$E(1) = 1, \qquad E(2) = 2, \qquad E(3) = 3, \qquad E(6) = 6.$$

For every non-terminal value, the following holds recursively

$$E(N) = E(F(N)).$$

The terminal value function is therefore invariant along every descent path. If

$$N_0 \to N_1 \to \cdots \to N_r$$

is a complete descent path, then

$$E(N_0) = E(N_1) = \cdots = E(N_r) = N_r.$$

The terminal value is therefore not assigned solely to the initial value. It is a common property of all nodes on the same descent path.

6.2 Definition of the triatrad classes

For every terminal element

$$e \in \mathcal{T}$$

, the corresponding triatrad class is defined by

$$\mathcal{C}_e = \{N \in \mathbb{N}_{>0}: E(N) = e\}$$

.

This results in the four classes

$$\mathcal{C}_1, \qquad \mathcal{C}_2, \qquad \mathcal{C}_3, \qquad \mathcal{C}_6.$$

Each class contains its own terminal value:

$$1 \in \mathcal{C}_1, \qquad 2 \in \mathcal{C}_2, \qquad 3 \in \mathcal{C}_3, \qquad 6 \in \mathcal{C}_6.$$

A value belongs to$\mathcal{C}_e$ if and only if its complete descent path ends at$e$.

6.3 Disjointness of the classes

Let$e_1, e_2 \in \mathcal{T}$ be defined by

$$e_1 \neq e_2.$$

Suppose there exists a value

$$N \in \mathcal{C}_{e_1} \cap \mathcal{C}_{e_2}.$$

Then the following must hold simultaneously

$$E(N) = e_1$$

and

$$E(N) = e_2$$

hold simultaneously.

This contradicts the uniqueness of the limit function. Therefore,

$$\mathcal{C}_{e_1} \cap \mathcal{C}_{e_2} = \varnothing.$$

The four triatrad classes are thus pairwise disjoint.

6.4 Completeness of the class decomposition

For every

$$N \in \mathbb{N}_{>0}$$

the following holds by the Triatrad theorem

$$E(N) \in \{1,2,3,6\}.$$

Thus, every positive natural N belongs to at least one of the four classes. Because the classes are disjoint, it belongs to at most one class.
It follows that the decomposition is complete

$$\mathbb{N}_{>0} = \mathcal{C}_1 \dot{\cup} \mathcal{C}_2 \dot{\cup} \mathcal{C}_3 \dot{\cup} \mathcal{C}_6.$$

The symbol $\dot{\cup}$ denotes the disjoint union. The set of positive natural numbers is thus completely and uniquely partitioned into four terminal classes by the recursive triangular descent.

6.5 Four-Class Theorem

Theorem 3. The limit function E generates a complete disjoint partition of the positive natural numbers:

$$\mathbb{N}_{>0} = \mathcal{C}_1 \dot{\cup} \mathcal{C}_2 \dot{\cup} \mathcal{C}_3 \dot{\cup} \mathcal{C}_6.$$

Proof. By Theorem 2, for every $N \in \mathbb{N}_{>0}$, $E(N)$ takes exactly one value from $\mathcal{T} = \{1,2,3,6\}$. Therefore, every N belongs to at least one of the four classes. Due to the uniqueness of $E(N)$, no N can belong to two different classes. Thus, the classes are complete and pairwise disjoint.

6.6 First elements of the four classes

The first non-terminal examples already show that all four classes are non-empty:

$$4 \to 1,$$

$$5 \to 2,$$

$$9 \to 3,$$

$$16 \to 6.$$

It follows that

$$4 \in \mathcal{C}_1, \qquad 5 \in \mathcal{C}_2, \qquad 9 \in \mathcal{C}_3, \qquad 16 \in \mathcal{C}_6.$$

Further values are generated by backtracking. Any value that, in a single step, descends to an already known node of the class $\mathcal{C}_e$ also belongs to $\mathcal{C}_e$.

6.7 Recursive decomposition of triangular numbers

Let

$$N_0 \to N_1 \to \cdots \to N_r = E(N_0)$$

a complete descent path. For each step, the following holds

$$N_{k+1} = N_k - T_{t(N_k)}.$$

It follows that

$$N_k = N_{k+1} + T_{t(N_k)}.$$

By repeatedly substituting, we obtain

$$N_0 = E(N_0) + \sum_{k=0}^{r-1} T_{t(N_k)}.$$

Every positive natural number thus has a representation, uniquely determined by its descent path, as a terminal triatrad value plus a finite sum of adaptively selected triangular numbers:

$$N = E(N) + \sum_{k=0}^{r-1} \frac{t(N_k)(t(N_k)+1)}{2}.$$

This representation is not equivalent to an arbitrary decomposition into triangular numbers. The indices are not chosen freely, but are uniquely determined at each step from the current residual value.

The entire descent sequence therefore defines a unique descent word

$$t(N_0), t(N_1), \ldots, t(N_{r-1})$$

. This word contains the complete sequence of the removed triangular numbers and ends with exactly one triatrad value.

6.8 Invariance along converging paths

Let$N$ and$M$ be two positive natural numbers whose descent paths reach a common node$R$. Then there exist natural numbers$i$ and$j$ such that

$$F^i(N) = R$$

and

$$F^j(M) = R.$$

From this node onwards, both paths are identical. Consequently, the following holds

$$E(N) = E(R) = E(M).$$

Converging descent paths can therefore never belong to different triatrad classes. As soon as two paths share a common node, all upstream predecessors of both paths belong to the same terminal tree.

7. The four-tree decomposition

7.1 Directed descent graph

The recursive descent of triangular numbers can be represented as a directed graph. The set of vertices is

$$V = \mathbb{N}_{>0}.$$

For every non-terminal value N , a directed edge

$$N \to F(N)$$

is inserted.

Terminal values have no outgoing edge within the positive descent graph, as their formal successor is not positive.

Every non-terminal node has exactly one outgoing edge. The number of incoming edges, on the other hand, can be zero, one or greater than one. The graph is therefore deterministic in the forward direction, but branched in the reverse direction.

7.2 The four roots

According to the Triatrad Theorem, every directed path terminates at exactly one of the four terminal values

$$1, \quad 2, \quad 3, \quad 6.$$

These four vertices form the roots of the four backward trees.
The tree with root$e \in \mathcal{T}$ has the following set of nodes

$$\mathcal{C}_e.$$

The edges within this tree are precisely the descent relations

$$N \to F(N)$$

for all non-terminal

$$N \in \mathcal{C}_e.$$

7.3 Tree structure of a triatrad class

Each class$\mathcal{C}_e$ forms a directed tree oriented towards the root$e$.
Three properties are key to this:

Firstly, every node apart from the root has exactly one directed path to the root.

Secondly, there is no directed cycle within the class.

Thirdly, a node may have several immediate predecessors.

The direction of the operator points from larger to smaller numbers. If one considers the structure in reverse, a single node gives rise to all the values that descend to it in a single step. This results in branches.

7.4 Predecessors of a node

Let

$$M \in \mathbb{N}_{>0}$$

a fixed node. We are looking for all values$N$ for which

$$F(N) = M$$

holds.
Let

$$m = t(N).$$

From

$$F(N) = N - T_m = M$$

follows

$$N = M + T_m.$$

For such a candidate to actually have the index$m$, it must lie within the intervalI_m :

$$m(m-1) \leq M + T_m < m(m+1).$$

Where

$$T_m = \frac{m(m+1)}{2}$$

it follows that

$$m(m-1) \leq M + \frac{m(m+1)}{2} < m(m+1).$$

The left-hand inequality yields

$$M \geq m(m-1) - \frac{m(m+1)}{2}$$

and thus

$$M \geq \frac{m(m-3)}{2}.$$

From the right-hand inequality it follows

$$M < m(m+1) - \frac{m(m+1)}{2}$$

and thus

$$M < \frac{m(m+1)}{2}.$$

Since$M$ is an integer, this can be written as

$$M \leq \frac{m(m+1)}{2} - 1.$$

A value $N = M + T_m$ is therefore an immediate predecessor of$M$ if and only if

$$\frac{m(m-3)}{2} \leq M \leq \frac{m(m+1)}{2} - 1$$

holds.

7.5 Predecessor formula

For every fixed node$M$ and every positive integer index$m$ that satisfies the condition

$$\frac{m(m-3)}{2} \leq M \leq \frac{m(m+1)}{2} - 1$$

, there exists exactly one immediate predecessor

$$N = M + T_m.$$

Thus, the predecessor set of$M$ is:

$$\mathrm{Pred}(M) = \left\{ M + \frac{m(m+1)}{2} : m \in \mathbb{N}_{>0},\ \frac{m(m-3)}{2} \leq M \leq \frac{m(m+1)}{2} - 1 \right\}.$$

This formula enables the complete back-tracking construction of all four triatrad trees.

7.6 Finiteness of the set of immediate predecessors

For a fixed value$M$, the predecessor condition can only be satisfied by a finite number of indices$m$. From

$$\frac{m(m-3)}{2} \leq M$$

it follows that

$$m^2 - 3m - 2M \leq 0.$$

The positive root of the corresponding quadratic equation is

$$m = \frac{3 + \sqrt{9 + 8M}}{2}.$$

Therefore, for every possible predecessor index

$$m \leq \frac{3 + \sqrt{9 + 8M}}{2}.$$

Each node thus has only a finite number of immediate predecessors. The number of possible predecessors may increase as$M$ increases, but remains finite for each individual node.

7.7 Example of a backward construction

For the root

$$M = 1$$

the predecessor condition is

$$\frac{m(m-3)}{2} \leq 1 \leq \frac{m(m+1)}{2} - 1.$$

For$m = 2$, the following holds

$$-1 \leq 1 \leq 2.$$

This gives the predecessor

$$N = 1 + T_2 = 1 + 3 = 4.$$

In fact, the following holds

$$F(4) = 1.$$

For$m = 3$, the following holds

$$0 \leq 1 \leq 5.$$

From this, the predecessor follows

$$N = 1 + T_3 = 7.$$

Here too, the following holds

$$F(7) = 1.$$

For $m = 4$, the following holds

$$2 \leq 1,$$

which is incorrect. Larger values of m also do not satisfy the left-hand inequality.
1 therefore has exactly the following immediate non-terminal predecessors

$$4 \quad \text{und} \quad 7.$$

The first segment of the tree $\mathcal{C}_1$ thus begins with

$$4 \to 1$$

and

$$7 \to 1.$$

From the nodes 4 and 7 , further predecessor layers can be generated recursively.

7.8 Merging of descent paths

If two different values, N_1 and N_2 , have the same successor M , then the following applies

$$N_1 \to M$$

and

$$N_2 \to M.$$

From the node M onwards, the two descent paths are identical. The paths merge and cannot separate again during the subsequent descent, as each node has only one successor.
The forward-directed graph is thus characterised by progressive merging. Viewed in reverse, the same structure corresponds to a branching.

7.9 Infinity of the four trees

Each of the four terminal trees has an infinite number of nodes.

Let$M$ be an arbitrary node. For sufficiently large suitable predecessor chains, values can be constructed that, after a finite number of steps, descend to$M$. Even the repeated application of the predecessor formula generates ever-larger values.

Since every root has at least one non-terminal predecessor and the backward construction can be iterated, values of unbounded magnitude arise in every class. Thus, all four sets are

$$\mathcal{C}_1, \quad \mathcal{C}_2, \quad \mathcal{C}_3, \quad \mathcal{C}_6$$

are infinite.

This infinity does not imply that the four classes must be equally densely distributed. The tree sizes within finite intervals may differ considerably from one another. Even large local gaps in individual classes are compatible with global infinity.

7.10 Four-Tree Theorem

Theorem 4. The directed descent graph of the recursive triangular number operator decomposes into exactly four disjoint directed back-tracing trees with roots 1 ,2 ,3 and 6 .

Proof. Every positive natural number has exactly one finite directed path to exactly one terminal value. By Theorem 2, this terminal value is one of the four elements1 ,2 ,3 or6 . Due to the uniqueness of the terminal value, the four sets of nodes are pairwise disjoint. There are no cycles. Every non-terminal node has exactly one successor, but may have several predecessors. Consequently, each terminal value class forms a back-tree directed towards the terminal root. Together, these four trees cover the entire set$\mathbb{N}_{>0}$.

7.11 The Triatrad Forest

The collection of the four trees can be referred to as a directed forest

$$\mathfrak{F}_{\mathcal{T}} = \mathfrak{B}_1 \dot{\cup} \mathfrak{B}_2 \dot{\cup} \mathfrak{B}_3 \dot{\cup} \mathfrak{B}_6$$

.

$\mathfrak{B}_e$ has the set of nodes$\mathcal{C}_e$ and the root$e$.

The four-tree decomposition is thus more robust than the mere statement that four terminal values exist. It assigns to every positive natural number both a terminal identity and a unique position within one of four global back-trees.

8. The problem of direct class assignment

8.1 Recursive determinability

The terminal value function is algorithmically completely determined. For terminal values, the following holds

$$E(1) = 1, \qquad E(2) = 2, \qquad E(3) = 3, \qquad E(6) = 6.$$

For all other values, the following holds

$$E(N) = E(F(N)).$$

Thus, the terminal value of any positive natural number can be computed by repeatedly applying the operator.
However, recursive determinability does not resolve the question of which arithmetic property determines the limit.

8.2 Direct Determination

We are seeking a function or decision rule that determines the value

$$E(N) \in \{1,2,3,6\}$$

without having to execute the descent sequence in full.
Such a rule would take the form

$$E(N) = G(N),$$

where G does not rely recursively on the repeated application of$F$.
A direct rule could, for example, be based on a congruence structure, a particular representation of$N$, an invariant, a digit or bit structure, or a closed characterisation of the back trees. To date, no such general rule is known.

8.3 Distinguishing between two questions

The Triatrad Theorem answers the question

Welche terminalen Werte sind möglich?

in full.
The answer is

$$\{1,2,3,6\}.$$

This is to be distinguished from the question

Welcher dieser vier Werte gehört zu einem gegebenen N?

This second question remains open.

The completeness of the final set is therefore no substitute for a direct class characterisation. Conversely, a direct class rule would not replace the triatrad theorem, but rather complement it.

8.4 Sought-after class invariant

A particularly robust solution to the classification problem would be an arithmetic quantity

$$I(N)$$

with the property that the value of$I(N)$ uniquely determines the class of$N$.

Then there could exist a mapping

$$\Phi$$

such that

$$E(N) = \Phi(I(N)).$$

Such a quantity would not necessarily have to remain numerically constant during the descent. It would suffice if it already fully encoded the terminal membership in the initial value.

An even stronger form would be an invariant with

$$I(N) = I(F(N))$$

for all non-terminal$N$. Then$I$ would be constant along every descent path and could be directly linked to the terminal value.

8.5 Recursive descent word as complete encoding

Even without a direct invariant, every value already possesses a complete recursive encoding via its descent word

$$W(N) = (t(N_0), t(N_1), \dots, t(N_{r-1})).$$

This word determines all removed triangular numbers and thus the entire path. It consequently also contains the final value.
The open problem is not whether sufficient information is available. By definition, the entire sequence of paths contains the terminal value. Rather, what is sought is a significantly more compact property that can be calculated directly from$N$ and yields the same class assignment.

8.6 The role of discrete deviations

Since

$$F(N) \approx \frac{N}{2}$$

holds, the descent paths of all four classes have a similar rough size envelope. The class information cannot therefore be derived solely from the approximate halving.
Let

$$N = m(m-1) + r$$

with

$$m = t(N)$$

and

$$0 \leq r \leq 2m - 1.$$

Then the following holds

$$F(N) = \frac{m(m-3)}{2} + r.$$

Compared with an exact halving, the deviation can be described by

$$\delta(N) = F(N) - \frac{N}{2}$$

be described.
Substituting the interval form gives

$$\delta(N) = \frac{r}{2} - m.$$

Thus, the exact step is

$$F(N) = \frac{N}{2} + \frac{r}{2} - m.$$

Since

$$0 \leq r \leq 2m - 1$$

the following always holds

$$\frac{r}{2} - m \leq -\frac{1}{2}.$$

It follows that

$$F(N) < \frac{N}{2}.$$

The coarse dynamics are determined by the halving term. The finer path structure lies in the sequence of discrete corrections

$$\frac{r_k}{2} - m_k.$$

It is possible that the terminal class is determined by a hitherto unknown overall structure of this sequence of corrections.

8.7 Direct class assignment as an independent research problem

The search for a direct limit function is not merely a question of acceleration. Such a function could reveal a property of natural numbers which, whilst reliably identified by recursion, has not yet been given an arithmetic name.
The key open question is therefore:

Was ist $E(N)$ als arithmetische Funktion?

This gives rise, in particular, to the following questions:
Do the four classes possess different congruence signatures?

Do number forms exist that occur exclusively or predominantly in a particular class?

Can class membership be determined from the interval position$r$ or from a sequence of such positions?

Does every tree have a closed backward description?

Is there a non-recursive representation of$E(N)$?

These questions no longer concern the existence of the Triatrad, but rather the intrinsic arithmetic significance of the four-tree decomposition.

8.8 Relationship between proof and empirical evidence

The canonical triatrad theorem has been proved independently of numerical test sequences. The empirical investigations therefore do not serve to demonstrate that the set of limit values

$$\mathcal{T} = \{1,2,3,6\}$$

is complete for all positive natural starting values. This statement already follows from the definition of the operator and the characterisation of the terminal values.
The numerical investigation focuses on a different level. It no longer asks which terminal values are possible, but rather how the four terminal value classes are arranged within large contiguous ranges of numbers. In particular, the investigation examines the global class weights, the order of neighbouring terminal values, local shifts in density, sequences of identical classes, and extended intervals in which a single class is completely absent.

Thus, proof and empirical analysis complement one another. The proof determines the complete terminal set. The empirical analysis examines the internal spatial structure of the resulting Vierbaum decomposition.

Formally, it is already known for every$N \in \mathbb{N}_{>0}$ that

$$E(N) \in \{1,2,3,6\}$$

holds. The statistical investigation, on the other hand, focuses on functions and relations of the form

$$C_e(A,B) = \#\{N \in [A,B] : E(N) = e\},$$

$$P_e(A,B) = \frac{C_e(A,B)}{B - A + 1},$$

as well as on the neighbourhood relation

$$E(N) \to E(N+1).$$

For a terminal value$e \in \mathcal{T}$, it is also investigated whether, and over what length, intervals exist in which

$$E(N) \neq e$$

holds for all values contained therein.
The empirical results cannot replace a direct arithmetic classification rule. However, they can provide indications as to whether the four classes are distributed

approximately independently or whether their arrangement is characterised by additional deterministic structures. In particular, large-scale voids, directed transitions and recurring local distribution states would be difficult to reconcile with a purely unstructured four-class sequence.

8.9 Transition to numerical analysis

The central empirical question is:

Wie sind die vier Triatrad-Klassen entlang der natürlichen Zahlen angeordnet?

This question has several dimensions. Firstly, it is necessary to determine the proportion occupied by each class within a large contiguous region. Next, it must be investigated whether the terminal values of neighbouring numbers behave independently of one another or form preferred transitions. Finally, it must be examined whether the classes are locally represented everywhere or whether contiguous exclusion regions of individual terminal values occur.

For these purposes, the terminal value function was calculated in the range

$$10 \leq N \leq 1.000.000.009$$

for exactly one billion consecutive natural numbers. The resulting dataset contains, for each examined$N$, only the unique assignment

$$N \mapsto E(N).$$

The individual descent paths were not recorded in this test series. This made it possible to limit the data set to the information essential for the class and void analysis. The calculation of descent times and complete routes was carried out independently of this, using smaller test series and selected individual values.

9. Methodology of the numerical investigation

9.1 Objective of the main test series

The aim of the main test series was the comprehensive determination of the Triatrad final value for one billion consecutive initial values. The range investigated began at

$$N_{\mathrm{Start}} = 10$$

and had a length of

$$L = 1.000.000.000.$$

The last value included was therefore

$$N_{\mathrm{Ende}} = N_{\mathrm{Start}} + L - 1 = 1.000.000.009.$$

For each$N$, the recursive triangular number descent was carried out until one of the four terminal values was reached. Subsequently, only

$$E(N) \in \{1,2,3,6\}.$$

This reduction to the terminal value was necessary because storing all intermediate stages would have significantly increased the volume of data. Even for smaller test sequences, it became apparent that the number of stored stages per initial value increases with its magnitude. However, the complete paths are not required for the investigation of class weights, sequences and voids.

9.2 Integer calculation of the descent

The index function was calculated without floating-point arithmetic. For a current value N , the following applies

$$t(N) = \left\lfloor \frac{\sqrt{4N+1}+1}{2} \right\rfloor.$$

The programme used the integer square root, thereby eliminating rounding errors caused by binary floating-point numbers. The next descent value was determined by

$$N_{\mathrm{neu}} = N - \frac{t(N)(t(N)+1)}{2}$$

.

The recursion ended as soon as the current value

$$\{1,2,3,6\}$$

. For the main test series, a high maximum number of steps was also set as a safety limit. This limit was not reached in any of the calculated cases.

9.3 Compact final-value encoding

As only four different final values can occur, each final value can be represented by exactly two bits. The following encoding was used

$$1 \mapsto 00,$$

$$2 \mapsto 01,$$

$$3 \mapsto 10,$$

$$6 \mapsto 11.$$

This allowed four consecutive final values to be stored in a single byte. For one billion mappings, this results in a net payload of

$$1.000.000.000 \cdot \frac{2}{8} = 250.000.000$$

bytes.

The two-bit codes were assigned within each byte in a fixed order, starting with the most significant bit positions. This means that every stored value can be uniquely reconstructed from its position within the overall sequence.

For the index$i$, where

$$0 \leq i < L$$

the stored final value corresponds to the initial value

$$N = N_{\text{Start}} + i.$$

The database therefore did not need to store each input value again as a separate integer. Its position in the continuous sequence was sufficient.

9.4 Chunk structure of the database

The billion final values were stored in blocks of

$$1.000.000$$

values. This resulted in exactly

$$1.000$$

chunks.

Each complete chunk contained

$$1.000.000 \cdot 2 = 2.000.000$$

bits and thus

$$250.000$$

bytes of payload data.

For each chunk, at least its sequence number, the start index, the number of end values it contained and the packed binary block were stored. The chunk numbers ranged from

$$0$$

to

$$999.$$

The corresponding start indices were

$$0, \quad 1.000.000, \quad 2.000.000, \quad \ldots, \quad 999.000.000.$$

The storage of a chunk and the updating of the associated progress status were carried out simultaneously. This prevented a chunk from being considered stored without the metadata reflecting the same state, or vice versa.

9.5 Resumability of the calculation run

The block-by-block structure allowed the process to be resumed following an interruption. Upon restarting, the system checked how many complete chunks were already available. The calculation could then be resumed at the first input value that had not yet been saved.

In the event of an unexpected termination, at most the values of the chunk that was currently still incomplete could be lost. Chunks that had already been fully saved remained unchanged. This design was important for a long-running calculation process, as it prevented the need for a complete restart of the billion-test series.

9.6 Metadata

In addition to the binary blocks, metadata relating to the entire test series was stored. This included, in particular,

$$N_{\text{Start}},$$

$$N_{\text{Ende}},$$

$$L,$$

the chunk size, the number of values already calculated, the completeness status, the encoding version and the four cumulative class counters.

The sum of the four final-value counters had to match the number of stored mappings at all times:

$$A_1 + A_2 + A_3 + A_6 = L_{\text{berechnet}}.$$

For the completed run, the following applied accordingly

$$A_1 + A_2 + A_3 + A_6 = 1.000.000.000.$$

9.7 Separation of Raw Data and Analysis

The SQLite database served exclusively as a compact primary source of the complete sequence of final values. All statistical tables were generated by standalone evaluation scripts once the calculation had been completed.

This separation offers several advantages. The raw data set remains unchanged. New research questions can be investigated without having to recalculate the descent analyses. Different analytical methods draw on the same validated decoding. Furthermore, local window sizes, void thresholds and pattern lengths can be modified retrospectively.

The database itself is not provided as part of the publication. The machine-readable analysis tables generated from it are made available collectively as a compressed supplementary dataset for the Zenodo publication, named ‘ ’. This supplementary dataset comprises the CSV analysis files used for the paper. The programmes used for generation and analysis are also available on Zenodo as a ZIP file.

9.8 Validation of the database

Prior to statistical interpretation, the entire dataset was checked for technical accuracy and content. The validation comprised several independent stages.
Firstly, the metadata was checked. This involved verifying

$$N_{\text{Ende}} = N_{\text{Start}} + L - 1$$

the number of calculated values corresponds to the specified length and whether the database is marked as complete.
The chunk structure was then checked. The database had to contain exactly 1.000 chunks. The chunk numbers and start indices had to follow on from one another without any gaps. Neither missing nor duplicate nor overlapping chunks were permitted.

For each chunk, a check was also carried out to ensure that the stored number of end values was consistent with the size of the binary block. In general, the following had to apply:

$$\text{Länge(BLOB)} = \left\lceil \frac{\text{Anzahl der Werte}}{4} \right\rceil.$$

In addition, the unused padding bits of the last byte were checked in cases where a chunk did not contain a number of values divisible by four. In the present main test series, all chunks were complete and contained exactly one million values.

9.9 Complete Recount

The four class frequencies were not only counted during the original calculation. Upon completion, all one billion two-bit values were decoded again from the stored binary blocks and counted in full.
The recount yielded:

$$A_1 = 293.421.555,$$

$$A_2 = 393.924.939,$$

$$A_3 = 194.130.342,$$

$$A_6 = 118.523.164.$$

These four values each corresponded exactly to the counters stored in the metadata. Their sum also equalled the intended total exactly:

$$293.421.555 + 393.924.939 + 194.130.342 + 118.523.164 = 1.000.000.000.$$

This confirmed that all stored two-bit codes could be uniquely decoded and that the complete final sequence contained neither missing nor additional entries.

9.10 Direct mathematical sample testing

The technical consistency of the binary data alone is not sufficient to confirm that the stored final values were calculated mathematically correctly. For this reason, a direct recalculation of selected input values was also carried out.
The sample comprised a total of

$$135.988$$

various input values. These included values at the beginning and end of the total range, values distributed evenly across the billion, randomly selected values, and values immediately at and adjacent to chunk boundaries.
For each selected 'N', the complete triangular number descent was carried out again. The newly calculated final value was then compared with the two-bit value decoded from the database.

No discrepancies were found across the entire sample:

$$\#\{N: E_{\mathrm{neu}}(N) \neq E_{\mathrm{DB}}(N)\} = 0.$$

The validation thus confirms both the technical integrity and the mathematical plausibility of the database used. Whilst it does not replace formal verification of every single stored data record, together with the complete binary recount it forms a robust basis for the subsequent statistical analyses.

10. Global distribution of the Triatrad classes

10.1 Absolute class frequencies

For the entire range

$$10 \leq N \leq 1.000.000.009$$

, the following absolute frequencies were determined:

Triatrad final value	Number
1	293.421.555
2	393.924.939
3	194.130.342
6	118.523.164
Total	1.000.000.000

The class '$\mathcal{C}_2$ ' has the largest catchment area within the range under investigation. This is followed by '$\mathcal{C}_1$ ', '$\mathcal{C}_3$ ' and '$\mathcal{C}_6$ '.

The ranking is therefore

$$A_2 > A_1 > A_3 > A_6.$$

Globally, the class$\mathcal{C}_6$ is the rarest of the four end-value classes examined.

10.2 Relative class proportions

The relative proportions are calculated as follows

$$P_e = \frac{A_e}{1.000.000.000}.$$

This gives:

$$P_1 = 0{,}293421555 = 29{,}3421555\,\%,$$

$$P_2 = 0{,}393924939 = 39{,}3924939\,\%,$$

$$P_3 = 0{,}194130342 = 19{,}4130342\,\%,$$

$$P_6 = 0{,}118523164 = 11{,}8523164\,\%.$$

Rounded to six decimal places, the global distribution is:

Triatrad end value	Proportion
1	29,342156 %

Triatrad end value	Proportion
2	39,392494 %
3	19,413034 %
6	11,852316 %

The four classes are therefore weighted very unevenly. There is no equal distribution of$25\,\%$ in each case.

10.3 Comparison with a simple reference distribution

Even earlier preliminary investigations suggested a rough similarity to the reference sequence

$$30\,\%,\quad 40\,\%,\quad 20\,\%,\quad 10\,\%$$

for the classes

$$1,\quad 2,\quad 3,\quad 6$$

.

This reference corresponds to the ratio

$$A_1 : A_2 : A_3 : A_6 \approx 3 : 4 : 2 : 1.$$

For one billion values, this reference would yield the following figures:

Final value	Reference count	Measured count	Deviation
1	300.000.000	293.421.555	−6.578.445
2	400.000.000	393.924.939	−6.075.061
3	200.000.000	194.130.342	−5.869.658
6	100.000.000	118.523.164	+18.523.164

The negative deviations for the classes1 ,2 and3 add up to

$$6.578.445 + 6.075.061 + 5.869.658 = 18.523.164.$$

This value corresponds exactly to the excess of the class6 relative to the chosen reference.

The measured distribution is therefore close to

$$3 : 4 : 2 : 1,$$

but exhibits a systematic shift in favour of the class6 . This reference is, for the time being, merely a descriptive benchmark. No proven distribution law follows from it.

10.4 Summary of main and secondary classes

The two most frequent classes, ‘1 ’ and ‘2 ’, together account for the proportion

$$P_1 + P_2 = 29{,}3421555\,\% + 39{,}3924939\,\% = 68{,}7346494\,\%.$$

The classes3 and 6 together account for

$$P_3 + P_6 = 19{,}4130342\,\% + 11{,}8523164\,\% = 31{,}2653506\,\%.$$

Thus, in the range under investigation, more than two-thirds of all natural values are accounted for by the two trees with roots 1 and2 .

However, this summary must not be interpreted as evidence of an algebraically grounded dichotomy. It initially describes only the observed total weights. Furthermore, the subsequent local analyses show that these global proportions mask significant regional variations.

10.5 Global Stability and Local Variability

The cumulative distributions remained within the same rough order of magnitude across large parts of the investigated region during the calculation. In particular, the ranking

$$2 > 1 > 3 > 6$$

remained globally consistent.

However, an approximately stable cumulative distribution does not mean that every sub-section must have similar proportions. The overall values arise from the aggregation of very different local conditions.

For example, one area may be completely free of the ‘6 ’ class, whilst another area may have an above-average proportion of this class. In the overall cumulative distribution, such areas may partially cancel each other out.

The global frequency therefore answers the question:

Wie viele Werte gehören insgesamt zu jeder Klasse?

It does not answer:

Wie sind diese Werte entlang der Zahlenachse angeordnet?

This second question requires an examination of local block distributions, neighbouring end values and class-free intervals.

11. Order and Neighbourhood of Endpoints

11.1 The sequence of terminal values

The complete data set can be regarded as a sequence

$$\mathcal{E} = (E(10), E(11), E(12), \dots, E(1.000.000.009))$$

.

Each sequence element belongs to the alphabet

$$\{1,2,3,6\}.$$

The global frequency merely describes how often each symbol occurs in this sequence. To capture the internal arrangement, neighbouring pairs and longer class words must also be examined.

For two terminal values$a, b \in \mathcal{T}$, the transition number is defined by

$$M_{ab} = \#\{N : E(N) = a,\ E(N+1) = b\}.$$

The complete transition numbers form a matrix

$$M = (M_{ab})_{a,b \in \mathcal{T}}.$$

As there are four classes, this matrix contains a total of

$$4^2 = 16$$

possible directed transitions.

11.2 Directed transitions

The empirical analysis shows that the transitions cannot be explained by the global class weights alone. Certain transitions occur significantly more frequently than others. Furthermore, the directed relationships are generally not symmetrical.

This means that a high frequency of

$$a \to b$$

does not automatically imply a similarly high frequency of

$$b \to a$$

.

The Triatrad sequence thus possesses a directed local structure. The class of a value contains measurable information about the probable class of its immediate neighbour.

A statistically independent sequence with the same global frequencies would, for a transition$a \to b$, approximately yield the probability

$$P(E(N+1) = b \mid E(N) = a) \approx P_b$$

.

However, the observed transition probabilities sometimes deviate significantly from this reference. This shows that the terminal values of consecutive natural numbers cannot be described as independent draws from a fixed four-class distribution.

11.3 Functional differences between the four classes

The transition analysis suggests that the four classes play different roles within the local sequence.

The class2 frequently appears as a distribution or transition class. Notable transitions lead from it to both the class3 and the class6 .

The class ‘1 ’ appears particularly frequently as the immediate successor to the class ‘6 ’. The class ‘6 ’ therefore tends to return to the domain of the class ‘1 ’ after it appears.

By contrast, the classes3 and 6 show only a slight tendency to follow immediately after themselves. Long sequences of the same type comprising these two classes are therefore significantly less common than corresponding sequences of the classes1 or2 .

These differences show that the four terminal values are not merely four interchangeable labels. Their positions in the terminal value sequence each possess their own local transition characteristics.

11.4 Sequences of Identical End Values

For every $e \in \mathcal{T}$, a run length can be defined as the maximum sequence of consecutive values for which

$$E(N) = E(N+1) = \cdots = E(N+\ell-1) = e$$

holds.

The number ℓ is the length of this run.

The frequency and maximum length of such runs differ between the four classes. In particular, the low self-neighbourhood of the classes 3 and 6 limits their contiguous runs.

The analysis of run lengths complements the void analysis. A run describes a local concentration of a class. A void, by contrast, describes its complete absence.

This provides two opposing metrics:

maximale lokale Anwesenheit

and

maximale lokale Abwesenheit.

11.5 Short class words

In addition to pairs, sequences of lengths three and four were also examined. For a word length of k , there are theoretically

$$4^k$$

possible class words.
This results in

$$4^2 = 16,$$

$$4^3 = 64$$

and

$$4^4 = 256$$

possible words of lengths two, three and four.
The frequency distributions of these words are highly uneven. Certain sequences occur more frequently, whilst others may be rare or, within certain local regimes, completely excluded.

Particularly striking are cycle-like transition sequences of the forms

$$1 \to 2 \to 3 \to 1$$

and

$$1 \to 2 \to 6 \to 1.$$

These sequences should not be interpreted as periodic laws governing the entire limit sequence. However, they describe preferred local transition directions, which become apparent in the frequency of short-class words.

11.6 Significance of sequence analysis

The analysis of the sequence of final values provides an initial indication that the four-tree decomposition is not only globally unevenly weighted but is also organised locally. The measured sequence also possesses:

ungleiche Klassenanteile,

gerichtete Nachbarschaftsbeziehungen,

klassenspezifische Lauflängen

and

bevorzugte kurze Klassenwörter.

These properties contradict the notion that the final-value function merely generates unstructured four-valued noise along the natural numbers. The strongest deviation from such a notion, however, is evident in the extended class-free intervals of the class6 . These are examined in the following chapter.

12. The 6-Voids

12.1 Definition of a class void

Let $e \in \mathcal{T}$ be a triatrad end value. An e void is a maximal contiguous interval

$$[a, b] \subset \mathbb{N}_{>0},$$

such that

$$E(N) \neq e$$

for which, for all

$$a \leq N \leq b$$

holds.

Maximality means that, immediately before and after the interval – provided these values lie within the total range under consideration – a representative of the class e appears again:

$$E(a-1) = e$$

and

$$E(b+1) = e.$$

The length of the void is defined by

$$V_e(a, b) = b - a + 1.$$

A void therefore does not measure the global rarity of a class, but rather its maximum contiguous absence within a specific range of numbers.

This distinction is crucial. The global frequency

$$P_e(A, B) = \frac{\#\{N \in [A, B] : E(N) = e\}}{B - A + 1}$$

answers the question of how many values within an interval belong to the class 'e'. The void length, on the other hand, answers the question of how these values are spatially arranged along the number line.

Two sequences of end values may have the same global class frequencies yet exhibit completely different void structures. A class distributed regularly may only have short gaps. The same number of class occurrences can produce very long class-free intervals if they are heavily clustered.

12.2 Voids as a measure of structure

For a class$e$ and an upper limit of analysis$X$, the maximum void length can be described by

$$M_e(X) = \max\{V_e(a,b): 1 \le a \le b \le X\}$$

.

The quantity$M_e(X)$ is an independent parameter of the sequence of limit values. It cannot be derived directly from the total proportion$P_e(1,X)$.

For a nearly independent random sequence with class probabilityp_e , the probability of an$e$ -free interval of length$L$ would be approximately

$$(1-p_e)^L$$

.

For the class6 , a global proportion of

$$p_6 = 0{,}118523164$$

was measured within the billion-test series. In an independent distribution with this proportion, even voids of a few hundred positions would be exceptionally rare. The 6-voids actually observed, by contrast, reach lengths ranging from several million to over 34 million consecutive natural numbers.

The void structure of the class6 cannot therefore be understood as a direct consequence of its lower global frequency.

12.3 First large-scale 6-void

An early sub-study of the range

$$100.000.000 \le N \le 100.999.999$$

did not reveal a single value ending in6 .

The distribution of this million-block was as follows:

$$E(N) = 1: 369.438,$$
$$E(N) = 2: 328.977,$$
$$E(N) = 3: 301.585,$$
$$E(N) = 6: 0.$$

This resulted in the following proportions

$$P_1 = 36{,}9438\,\%,$$
$$P_2 = 32{,}8977\,\%,$$
$$P_3 = 30{,}1585\,\%,$$
$$P_6 = 0\,\%.$$

The subsequent precise void analysis showed that this million-block constituted only part of a considerably larger contiguous 6-void interval.

The last value before this section with

$$E(N) = 6$$

was

$$97.947.567.$$

The next value with an end value of6 was

$$102.234.133.$$

Thus, the maximum 6-free region consisted of

$$97.947.568 \leq N \leq 102.234.132.$$

Its length was

$$102.234.132 - 97.947.568 + 1 = 4.286.565.$$

This finding was the first clear indication that the class6 possesses large-scale exclusion zones.

12.4 The ten largest 6-voids

A complete analysis of the billion-node database revealed the following ten largest 6-voids:

Rank	Start of the void	Void length
1	783.765.283	34.147.153
2	391.863.277	17.083.459
3	195.912.571	8.559.412
4	97.947.568	4.286.565
5	48.967.315	2.145.602
6	24.477.814	1.077.428
7	12.233.988	542.214
8	6.114.237	273.440
9	3.055.136	138.413

Rank	Start of the void	Void length
10	1.526.510	69.461

The lengths do not form an irregular sequence of independent extreme values. They exhibit a striking pattern of approximately doubling.

For the largest voids, for example:

$$34.147.153 \approx 2 \cdot 17.083.459,$$

$$17.083.459 \approx 2 \cdot 8.559.412,$$

$$8.559.412 \approx 2 \cdot 4.286.565,$$

$$4.286.565 \approx 2 \cdot 2.145.602.$$

The starting points of the voids also grow approximately by doubling:

$$1.526.510, \quad 3.055.136, \quad 6.114.237, \quad 12.233.988, \quad 24.477.814,$$

$$48.967.315, \quad 97.947.568, \quad 195.912.571, \quad 391.863.277, \quad 783.765.283.$$

This sequence suggests that the large 6-voids are not merely randomly occurring extreme gaps. They appear to belong to a scaling family of number ranges.

12.5 Approximate doubling sequence

Let us denote the large 6-voids, ordered by increasing size, by

$$V_1, V_2, \dots, V_{10},$$

whereV_1 is the smallest andV_{10} the largest of the voids shown.
Then, approximately,

$$V_{k+1} \approx 2V_k.$$

The quotients are not exactly equal to2 . However, over several consecutive orders of magnitude, they are very close to doubling.
The deviation from an exact power series is mathematically just as significant as the rough scaling. It shows that the void boundaries are not determined by a trivial scheme of the form

$$a_k = c \cdot 2^k$$

. Nevertheless, a doubling dynamic appears to determine the large-scale behaviour.
A possible descriptive form is therefore:

$$V_k = c_k 2^k,$$

where the correction factorsc_k vary only slowly.
Such a representation is not yet a derived formula. It describes the empirical structure of the ten largest of the one billion 6-voids observed.

12.6 Growth of void lengths

The largest observed void length

$$34.147.153$$

is approximately twice as large as the second-largest and approximately eight times as large as the void observed at around10^8

$$4.286.565.$$

At the same time, the void positions shift into numerical ranges that are approximately double those of the previous ones.
This finding raises an important open question:

Wachsen die maximalen 6-Voids mit N unbeschränkt?

The data available to date only extends to approximately

$$10^9.$$

It shows a clear trend of growth within this range. However, this does not yet provide proof of unlimited growth.
One possible asymptotic question is:

$$\limsup_{X\to\infty} M_6(X) = \infty?$$

A more significant question would be whether the maximum void lengths along certain sequences increase approximately in proportion to the position of their intervals.
For the$k$ th largest void, the ratio

$$R_k = \frac{V_k}{a_k}$$

be considered, wherea_k is the start of the void. A possible constancy or slow variation in this ratio would indicate a scaling geometric structure.

12.7 Distinction between frequency and void geometry

The class6 is

$$11{,}8523164\,\%$$

globally the rarest class. This fact explains why its average spacings may be greater than those of the more frequent classes.
However, it does not explain the observed order of magnitude of the voids.

In the case of an approximately independent distribution, the average distance between two hits in the ‘6 ’ class would be of the order of

$$\frac{1}{p_6} \approx 8{,}44$$

.
This contrasts with a maximum observed void of

$$34.147.153$$

.
The largest 6-void is thus more than four million times as long as the average hit spacing predicted from the global frequency.

This ratio is not a formal significance test, but it illustrates the order of magnitude of the deviation between global density and local arrangement.

The ‘6 ’ class is therefore not only rarer than the other classes; it also possesses a distinct spatial geometry.

12.8 Comparison with the other Triatrad classes

The classes ‘1 ’, ‘2 ’ and ‘3 ’ also contain voids. However, within the range examined, their maximum gaps remain many orders of magnitude smaller than those of the ‘6 ’ class.
Particularly striking is the comparison between the maximum voids of the classes3 and 6 . The analysis revealed, approximately,

$$V_6^{\max} \approx 1.138.238 \cdot V_3^{\max}.$$

The largest 6-void is thus more than a million times longer than the largest void in the class3 .

This difference cannot be explained solely by the fact that the class6 is slightly less common than the class3 . Their overall proportions differ only by a factor of

$$\frac{P_3}{P_6} = \frac{0{,}194130342}{0{,}118523164} \approx 1{,}64.$$

By contrast, the maximum void lengths differ by more than six orders of magnitude.
Thus, the class ‘6 ’ has a different void structure not only quantitatively but also qualitatively.

12.9 Voids as deterministic exclusion regions

Since the sequence of limit values follows entirely deterministically from$N$, every void is also generated deterministically.
A 6-void

$$[a, b]$$

means that for every value in the entire interval

$$E(N) \in \{1,2,3\}$$

holds.
Thus, within this region, a complete terminal tree of the four-tree decomposition is missing. The locally reachable terminal set is reduced from

$$\mathcal{T} = \{1,2,3,6\}$$

to

$$\mathcal{T}_{[a,b]} = \{1,2,3\}.$$

This statement does not concern the global possibility of the terminal value6 . It concerns the local position of its neighbourhood along the natural numbers.
A large 6-void can therefore be understood as a local exclusion region of the tree$\mathcal{C}_6$.

Whether these exclusion regions can be described by a simple arithmetic inequality, by certain interval mappings of the operator, or by a more distant backward tree structure remains an open question.

13. Distribution regime of the large 6-voids

13.1 The Void as more than just a zero zone

In the simplest case, a void could be an isolated interval in which a class suddenly ceases to appear, whilst the distribution of the remaining classes outside it remains largely unchanged.
However, analysis of the large 6-voids reveals a different picture. The distribution of the classes1 ,2 and3 begins to change even before the actual void. Within the void, a distinctly different distribution pattern emerges. After the last 6-free value, the original distribution does not return immediately.

The void thus comprises three dynamically distinguishable regions:

Vorzone,

Nullzone,

Nachzone.

These regions were examined using fixed boundary windows around the large 6-voids. Overlapping intervals were merged to ensure that no natural value was assigned to multiple regions.

13.2 Distribution outside the large 6-voids

Those parts of the overall range that belonged neither to a selected large void nor to its immediate peripheral zones were treated as lying outside.
This outer region comprised

$$928.539.060$$

values.
The following distribution was found there:

$$E(N) = 1: 266.868.055,$$
$$E(N) = 2: 371.022.383,$$
$$E(N) = 3: 172.183.032,$$
$$E(N) = 6: 118.465.590.$$

The relative proportions were:

$$P_1^{\text{außen}} = 28{,}740638547\,\%,$$
$$P_2^{\text{außen}} = 39{,}957649493\,\%,$$
$$P_3^{\text{außen}} = 18{,}543434457\,\%,$$
$$P_6^{\text{außen}} = 12{,}758277503\,\%.$$

These values form the empirical reference framework for the voids and peripheral zones.

Outside the large voids, the ‘2 ’ class is clearly dominant, at just under$40\,\%$. The ‘6 ’ class accounts for a share of around$12{,}76\,\%$ there, which is even higher than its global average in billions.

13.3 Distribution within the large 6-Voids

Within the large 6-voids examined, the following applies by definition

$$P_6^{\text{Void}} = 0.$$

However, the remaining mass is not distributed proportionally to their outer proportions across the classes1 ,2 and $3`.

Approximately the following distribution was determined within the voids:

$$P_1^{\text{Void}} \approx 37{,}24\,\%,$$
$$P_2^{\text{Void}} \approx 31{,}99\,\%,$$
$$P_3^{\text{Void}} \approx 30{,}77\,\%,$$
$$P_6^{\text{Void}} = 0\,\%.$$

Compared with the outer region, this results in approximately the following shifts:

$$\Delta P_1 \approx +8{,}50 \text{ Prozentpunkte},$$

$$\Delta P_2 \approx -7{,}97 \text{ Prozentpunkte},$$

$$\Delta P_3 \approx +12{,}23 \text{ Prozentpunkte},$$

$$\Delta P_6 \approx -12{,}76 \text{ Prozentpunkte}.$$

The missing class6 is therefore not distributed evenly across the three remaining classes. The class $3` experiences the greatest relative and absolute increase.

The class ‘3 ’ increases from

$$18{,}54\,\%$$

outside to approximately

$$30{,}77\,\%$$

within the voids.

As a result, its proportion increases by approximately two-thirds compared with the value outside the voids.

The class1 also grows significantly. The class2 , on the other hand, loses share despite the complete disappearance of the class6 .

The Void is therefore not merely a state

$$P_6 = 0.$$

It is a complete alternative distribution regime:

$$6 \downarrow 0, \qquad 3 \uparrow\uparrow, \qquad 1 \uparrow, \qquad 2 \downarrow.$$

13.4 The pre-zone

For the pre-zones of the large voids under investigation, a total of

$$1.529.899$$

values were analysed.

The distribution was as follows:

$$E(N) = 1: 545.778,$$
$$E(N) = 2: 495.311,$$
$$E(N) = 3: 467.726,$$
$$E(N) = 6: 21.084.$$

This resulted in the following proportions:

$$P_1^{\mathrm{vor}} = 35{,}674119664\,\%,$$
$$P_2^{\mathrm{vor}} = 32{,}375405174\,\%,$$
$$P_3^{\mathrm{vor}} = 30{,}572344972\,\%,$$
$$P_6^{\mathrm{vor}} = 1{,}378130190\,\%.$$

The pre-zone is therefore already very close to the inner void regime.

Although the ‘6 ’ class still exists, its share has fallen from around

$$12{,}76\,\%$$

outside to just

$$1{,}38\,\%$$

.

At the same time, the ‘3 ’ class already accounts for a proportion of

$$30{,}57\,\%,$$

that is almost equal to the inner Void's share.

The redistribution therefore does not begin only at the first value that is completely free of 6s. The Void has a genuine pre-zone in which its distribution state is already establishing itself.

13.5 The Post-zone

For the post-zones,

$$1.200.000$$

were evaluated.

The class frequencies were as follows:

$$E(N) = 1: 411.935,$$
$$E(N) = 2: 422.270,$$
$$E(N) = 3: 329.305,$$
$$E(N) = 6: 36.490.$$

The relative proportions were as follows:

$$P_1^{\text{nach}} = 34{,}327916667\,\%,$$
$$P_2^{\text{nach}} = 35{,}189166667\,\%,$$
$$P_3^{\text{nach}} = 27{,}442083333\,\%,$$
$$P_6^{\text{nach}} = 3{,}040833333\,\%.$$

The class '6 ' is present again after the void, but at around

$$3{,}04\,\%$$

still falls far short of the outer level of

$$12{,}76\,\%.$$

The class '3 ' also remains

$$27{,}44\,\%$$

remains significantly elevated.

The post-zone therefore shows that the void does not end dynamically at the last 6-free value. The class distribution requires an additional distance to approach the outer regime again.

13.6 Asymmetry of entry and exit

The pre- and post-zones are not mirror images of each other.

Before the void, the proportion of class 6

1,3781 %.

After the void, it is

3,0408 %.

The ‘6 ’ class thus initially returns more strongly immediately after the void than it was present immediately before the void.
The proportions of the other classes also exhibit different patterns of transition. Before the void, the proportion of the ‘3 ’ class is

30,57 %

already almost at the inner Void level. After the Void, it has

27,44 %

has already declined further towards the outer value.
The observed pattern can therefore be described, approximately, as

langsames Auslaufen der Klasse 6

as follows:

vollständiger Nullzone

and subsequently

beginnender, aber verzögerter Rückkehr.

The trend is thus asymmetrical.

13.7 Overview of regional distribution

The four regimes investigated can be summarised as follows:

Region	Class 1	Class 2	Class 3	Class 6
Outside	28,7406 %	39,9576 %	18,5434 %	12,7583 %
Pre-zone	35,6741 %	32,3754 %	30,5723 %	1,3781 %
Void	approx. 37,24 %	approx. 31,99 %	approx. 30,77 %	0 %
Post-zone	34,3279 %	35,1892 %	27,4421 %	3,0408 %

The table shows a continuous transition between two distinctly different states.

The outer regime is characterised by the dominance of the ‘2 ’ class:

$$2 > 1 > 3 > 6.$$

The void regime, by contrast, has the following ranking

$$1 > 2 \approx 3 > 6,$$

where the classes ‘2 ’ and ‘3 ’ are considerably closer together and the class ‘6 ’ is completely absent.

The pre- and post-zones form transitional states between these two regimes.

13.8 Redistribution of the missing class 6

It would be natural to assume that, upon the disappearance of the class6 , its proportion would mainly be transferred to the globally most frequent class2 .

The opposite is the case.

Within the voids, the ‘2 ’ class loses approximately

$$7{,}97$$

percentage points.

The ‘3 ’ class gains around

$$12{,}23$$

percentage points, thereby accounting for the largest share of the redistribution.

The ‘1 ’ class gains around

$$8{,}50$$

percentage points.

The sum of these two gains amounts to approximately

$$20{,}73$$

percentage points. This offsets the full loss of the ‘6 ’ class and the additional loss of the ‘2 ’ class.

Formally, the following holds approximately:

$$\Delta P_1 + \Delta P_3 = -\Delta P_2 - \Delta P_6.$$

The void dynamics are thus characterised by a coupled shift:

$$(2{,}6) \rightarrow (1{,}3).$$

This does not mean that individual values are immediately reallocated in pairs from one class to another. The equation describes the change in the local class weights of the entire regime.

13.9 Transition structure within the void regimes

Analysis of neighbouring endpoints shows that within the large voids, not only do the class proportions change, but so do the transition probabilities.
Outside the voids, preferred structures occur, which can be approximated by

$$1 \to 2 \to 3 \to 1$$

and

$$1 \to 2 \to 6 \to 1$$

.
In particular, in front of a large 6-void, the transition

$$2 \to 6$$

already drops sharply, even though the class ‘6 ’ has not yet completely disappeared. Within the void, this transition is, by definition, impossible. At the same time, the sequence

$$2 \to 3$$

gains in significance.
The local regime thus shifts more strongly towards

$$1 \to 3 \to 1$$

with additional inflow via

$$2 \to 3 \to 1.$$

After the Void, the class6 returns, though initially with a different transition structure to that in the outer region. The transition from6 to2 is temporarily more pronounced in the immediate post-zone, whilst the transition from2 to6 remains well below its outer level. This shows that the void represents not merely a density deficit, but an altered local transition system.

13.10 The void as an independent distribution regime

The combined results from the density, fringe zone and transition analyses lead to a more robust interpretation of the 6-Voids.
A large 6-void is not merely an interval in which, by chance, no value from the class $\mathcal{C}_6$ occurs.

It is a contiguous range of numbers with its own distribution and transition structure.

Its characteristic features are:

$$P_6 \to 0,$$

P_3 steigt stark,

P_1 steigt,

P_2 sinkt,

as well as a change in the preferred transitions between neighbouring end values.
The pre-zone shows that this regime begins to develop even before the void is complete. The post-zone shows that it only gradually subsides again after the void.

The large 6-voids therefore form independent local distribution regimes within the global Triatrad sequence.

14. In-depth analysis of class words

14.1 Concentration of the triplet patterns

The examination of all consecutive triplet windows confirms the directed order already evident from the transition matrix. Among the

$$4^3 = 64$$

possible triplet patterns, whilst all combinations occur at least once, their frequencies differ by several orders of magnitude.

The five most frequent triplet patterns are:

Rank	Triplet pattern	Frequency
1	$2 \to 3 \to 1$	107.311.563
2	$3 \to 1 \to 2$	104.719.119
3	$6 \to 1 \to 2$	98.777.411
4	$1 \to 2 \to 3$	97.032.739
5	$2 \to 6 \to 1$	88.489.725

Together, these five patterns account for approximately

$$49{,}63\,\%$$

of all triplets. The ten most common patterns together account for about

$$75{,}93\,\%.$$

three-quarters of all local sequences of length three are thus concentrated in just ten of the total of64 possible patterns. There is no approximately uniform distribution of triplets.

This high concentration confirms that the Triatrad sequence possesses a distinct local grammar. A large proportion of the sequence is described by a small number of preferred transition chains.

14.2 Two dominant basic loops

The most frequent three-word patterns can predominantly be assigned to two directed cycles:

$$1 \to 2 \to 3 \to 1$$

and

$$1 \to 2 \to 6 \to 1.$$

The first cycle generates the shifted triplets

$$1 \to 2 \to 3,$$

$$2 \to 3 \to 1$$

and

$$3 \to 1 \to 2.$$

The second cycle generates, accordingly,

$$1 \to 2 \to 6,$$

$$2 \to 6 \to 1$$

and

$$6 \to 1 \to 2.$$

Almost all of these six sequences of three are among the most common patterns in the entire billion-test series.

The basic local movement can therefore be represented schematically as

$$1 \to 2 \to \begin{cases} 3, \\ 6 \end{cases} \to 1.$$

Within this scheme, the class 2 acts as a branching point. From there, the sequence tends to lead either to the class 3 or to the class 6 . Both branches then tend to return to the class 1 , from which the class 2 frequently follows.

This description is not a transition rule that applies without exception. All three-candle patterns do occur in principle. However, it describes the statistically dominant local price movement pattern.

14.3 Rare three-wave patterns

The most common three-candle pattern

$$2 \to 3 \to 1$$

occurs

$$107.311.563$$

times.

The rarest three-ball pattern

$$3 \to 6 \to 3$$

, on the other hand, only

$$24$$

times.

The ratio is

$$\frac{107.311.563}{24} \approx 4.471.315.$$

The most common three-word pattern is therefore about$4{,}47$ million times more common than the rarest.

The five rarest three-word patterns include:

Three-word patterns	Frequency
$3 \to 6 \to 3$	24
$6 \to 3 \to 6$	45
$3 \to 3 \to 6$	263
$3 \to 6 \to 2$	1.062
$6 \to 3 \to 3$	1.104

All of these particularly rare patterns contain a direct, close connection between the classes3 and 6 .

This results in a striking contrast. In large-scale block distributions, the classes ‘3 ’ and ‘6 ’ appear to partially replace one another. In the large 6-voids, the proportion of the class ‘3 ’ in particular rises sharply. Locally, however, ‘3 ’ and ‘6 ’ are extremely rarely found immediately adjacent to one another.

The two classes are therefore linked on a large scale, but largely separate locally.

14.4 Repetition patterns

Pure repetition patterns also differ considerably between the four classes.

The following frequencies were determined for triplets:

$$1 \to 1 \to 1: \quad 3.654,$$

$$2 \to 2 \to 2: \quad 19.443.087,$$

$$3 \to 3 \to 3: \quad 2.430,$$

$$6 \to 6 \to 6: \quad 87.215.$$

The class2 thus exhibits a very strong tendency to form short sequences of the same type. The classes1 and3 , by contrast, occur only extremely rarely three times in immediate succession, despite their considerable global frequencies.

The class6 is also noteworthy. Although it is the rarest class overall and has very large voids, the pattern

$$6 \to 6 \to 6$$

occurs significantly more frequently than the corresponding runs of three in the classes1 and3 .

The class ‘6 ’ thus combines two seemingly contradictory properties:

extrem lange vollständige Abwesenheit

and

lokale Bündelung bei ihrem Auftreten.

Its low global frequency is therefore not due to a uniformly sparse distribution. Rather, areas of complete absence alternate with regions of increased local concentration.

14.5 Maximum run lengths

The longest observed sequences of identical terminal values were:

Class	Maximum run length
1	3
2	8
3	3
6	5

For the classes1 and3 , no run of four or more identical terminal values occurs within the entire billion.

The class2 , on the other hand, achieves sequences of up to eight consecutive values. This is consistent with its role as a class that occurs strongly locally and is partly self-binding.

Despite its large-scale voids, the class6 achieves local runs of up to five matches. This finding also supports the interpretation of a spatially clustered class.

14.6 Four-item patterns and forbidden sequences

For sequences of length four, there theoretically exist

$$4^4 = 256$$

possible class words.

Unlike the triplets, not all possible sequences of four occur. Within the complete billion-test series,

$$23$$

256 s of the theoretically possible four-element patterns have a frequency of zero.

This marks the first time that completely forbidden local sequences have been observed.

The proportion of unobserved four-word patterns amounts to

$$\frac{23}{256} \approx 8{,}984375\,\%.$$

Almost nine per cent of all theoretically possible four-word sequences are thus completely excluded in the range under investigation.

As the dataset comprises one billion consecutive initial values, the absence of these patterns cannot simply be explained as a consequence of an insufficiently large sample. The conclusion remains, for the time being, limited to the range investigated; however, it represents strong empirical evidence of hard local transition constraints.

14.7 Most Frequent Four-Word Patterns

The most common four-word patterns are:

Rank	Four-word pattern	Frequency
1	$2 \to 6 \to 1 \to 2$	81.769.958
2	$2 \to 3 \to 1 \to 2$	73.293.422
3	$6 \to 1 \to 2 \to 3$	72.765.882
4	$1 \to 2 \to 3 \to 1$	54.220.941
5	$1 \to 2 \to 6 \to 1$	53.227.702

The two basic loops, already derived from the three-note patterns, thus appear immediately below the most common complete four-note sequences:

$$1 \to 2 \to 3 \to 1$$

and

$$1 \to 2 \to 6 \to 1.$$

In addition, their shifted continuations occur particularly frequently:

$$2 \to 3 \to 1 \to 2,$$

$$2 \to 6 \to 1 \to 2$$

and

$$6 \to 1 \to 2 \to 3.$$

The four-step patterns therefore confirm that the two basic loops are not merely a coincidental overlap of frequent individual transitions. The complete loops themselves are among the dominant local words.

14.8 Local transition grammar

The totality of transitions, triplet patterns, quadruplet patterns and run lengths can be interpreted as an empirical transition grammar of the triatrad sequence.
This grammar does not possess a completely rigid rule of the form

$$a \to b$$

with only one permissible successor. All four classes can, in principle, occur in different contexts.
Nevertheless, the transitions are heavily weighted. Some sequences dominate, others are massively suppressed, and at length four, completely forbidden words appear for the first time.

The basic empirical structure is:

$$1 \to 2,$$

from there, preferably

$$2 \to 3$$

or

$$2 \to 6,$$

preferred afterwards

$$3 \to 1$$

or

$$6 \rightarrow 1.$$

The class ‘2 ’ thus functions as a local distributor. The class ‘1 ’ frequently acts as a return point. The classes ‘3 ’ and ‘6 ’ form two alternative branches which appear to be broadly interconnected, but which very rarely merge directly into one another locally.

15. Changes to the transition grammar in the 6-voids

15.1 Conditional transition analysis

The global transition analysis describes all neighbouring terminal values of the number range under investigation collectively. However, the large 6-voids show that the sequence of terminal values exhibits different local regimes.
Therefore, the transition frequencies were examined separately for the following ranges:

innerhalb großer 6-Voids,

unmittelbar vor den Voids,

unmittelbar nach den Voids

and

außerhalb der Void- und Randzonen.

For the purposes of this study, 6-free intervals of a specified minimum length or greater were defined as large voids. Transitions that crossed a region boundary were not counted as belonging to either of the two adjacent regions. Consequently, all regional transition probabilities were determined exclusively from pairs that lay entirely within the same region.

15.2 Formation of the bond $1 \leftrightarrow 3$

The most significant change within the large 6-voids concerns the transition

$$1 \to 3.$$

Outside the Voids, its conditional probability is approximately

$$P(1 \to 3 \mid \text{außen}) = 9{,}34\,\%.$$

Within the Voids, it rises to

$$P(1 \to 3 \mid \text{Void}) = 39{,}71\,\%.$$

The increase is therefore

$$30{,}37$$

percentage points.
Relative to the value outside the Voids, this corresponds roughly to a factor of

$$\frac{39{,}71}{9{,}34} \approx 4{,}25.$$

The reverse transition is even more pronounced

$$3 \to 1.$$

Outside the voids, its conditional probability is approximately

$$69{,}48\,\%.$$

Within the voids, it rises to

$$98{,}39\,\%.$$

Thus, within a large 6-void, almost every value in the class 3 is immediately followed by a value in the class 1 .

These two changes produce a pronounced local binding

$$1 \leftrightarrow 3,$$

particularly in the form of the sequence

$$1 \to 3 \to 1.$$

This binding is a defining feature of the void regime.

15.3 Deflection of the class 2

Within the voids, the transition

$$2 \to 6$$

impossible.

However, the class ‘2 ’ is not exclusively redirected to a single substitute class. Instead, the probability of

$$2 \to 3$$

from approximately

$$39{,}35\,\%$$

outside to

$$48{,}52\,\%$$

inside the Voids.

Also,

$$2 \to 1$$

increases. Its probability rises from approximately

$$10{,}32\,\%$$

to

$$19{,}41\,\%.$$

The missing branch

$$2 \to 6$$

is thus predominantly replaced by an increased allocation to the classes ‘3 ’ and ‘1 ’.
The local structure shifts from the globally dominant branching

$$2 \to \begin{cases} 3, \\ 6 \end{cases}$$

to a void structure, in which

$$2 \to 3$$

is clearly favoured, and the class `3 ` subsequently leads almost inevitably to the class `1 `.
Schematically, this results in:

$$2 \to 3 \to 1.$$

15.4 Replacement of the 6-branch

Outside the voids, the sequence

$$2 \to 6 \to 1$$

one of the two dominant local branches.
Within a 6-void, this branch is completely absent. However, its absence does not merely lead to a proportional strengthening of the other global branch

$$2 \to 3 \to 1.$$

In addition, the strong alternating bond forms

$$1 \to 3 \to 1.$$

The void regime can therefore be described, approximately, by two modes of motion:

$$2 \to 3 \to 1$$

and

$$1 \to 3 \to 1.$$

The class ‘3 ’ thus assumes, within the voids, part of the function that is distributed outside the voids amongst the separate classes ‘3 ’ and ‘6 ’.

15.5 Harbingers of the regime change

The change in the transition grammar begins even before the actual void.
In particular, the transition

$$2 \to 6$$

breaks down almost completely in the pre-zone, although individual instances of the class6 are still present there.
The class frequency alone already illustrates this process through the decline of ‘P_6 ’ to approximately

$$1{,}38\,\%.$$

However, the transition analysis also shows that it is not only the number of 6-hits that is decreasing. Their very local embedding within the dominant sequence

$$2 \to 6 \to 1$$

is being dissolved.
The pre-zone is therefore not merely a diluted form of the outer area. It already possesses essential grammatical properties of the subsequent void.

15.6 Return after the void

After the void, the class ‘6 ’ reappears. However, its return does not immediately restore the outer regime.
The transition

$$2 \to 6$$

is initially only approximate in the post-zone

$$8{,}26\,\%.$$

Outside the void regions, by contrast, it amounts to approximately

$$26{,}81\,\%.$$

The inflow from the ‘2 ’ class, which is essential for the global 6-branch, thus remains significantly weakened after the Void.
At the same time, the transition

$$6 \to 2$$

in the post-zone with approximately

$$16{,}87\,\%$$

, whilst its external value is only about

$$7{,}41\,\%$$

.
The returning ‘6 ’ class therefore initially has a different local connection. It is still reached relatively rarely from the ‘2 ’ class, but subsequently leads to the ‘2 ’ class with above-average frequency.

Before the Void, the path

$$2 \to 6$$

. After the Void, the reverse path is temporarily

$$6 \to 2$$

is temporarily more pronounced.
This directional asymmetry demonstrates once again that the exit from the Void is not a mirror image of its entry.

15.7 Regime map of the transitions

The regional transition analysis can be summarised in simplified form.
Outside the voids, the following dominates:

$$1 \to 2 \to \begin{cases} 3, \\ 6 \end{cases} \to 1.$$

Within the voids, however, the following dominates:

$$1 \to 3 \to 1,$$

supplemented by

$$2 \to 3 \to 1.$$

The second structure is already being prepared before the void. After the void, it dissolves only gradually.

Thus, the void does not merely concern the presence or absence of a single symbol. It alters the local grammar of the entire four-class sequence.

16. Reference model of an independent class sequence

16.1 Purpose of the comparison

The triatrad sequence is entirely deterministic. It is not a random sequence and is not interpreted as a stochastic process in this work.
Nevertheless, an independent random model can serve as a reference. It answers the question of what orders of magnitude of transitions and voids would be expected if the same global class frequencies were arranged randomly along the number line without any spatial constraints.

The reference model uses the measured probabilities

$$p_1 = 0{,}293421555,$$

$$p_2 = 0{,}393924939,$$

$$p_3 = 0{,}194130342,$$

$$p_6 = 0{,}118523164.$$

Assuming independent positions, the probability of a transition would be $a \to b$

$$P(a \to b) = p_a p_b.$$

The conditional probability of the subsequent value$b$ would be independent of the initial value$a$:

$$P(E(N+1) = b \mid E(N) = a) = p_b.$$

16.2 Deviation of transitions

The empirical sequence of terminal values does not satisfy this independence condition.
The conditional transition probabilities differ, in some cases drastically, from the global proportions. This is particularly evident within the void regimes, where, for example,

$$P(3 \to 1 \mid \text{Void}) \approx 98{,}39\,\%$$

holds true, whilst the regional proportion of the class1 is only approximately

$$37{,}24\,\%$$

.

The strong concentration on a small number of three- and four-word patterns is also incompatible with an independent arrangement. In an independent model, the word probabilities would be determined by the products of the individual probabilities. The empirical sequence, by contrast, shows frequency differences of several million to one, as well as a complete absence of four-word patterns.

16.3 Expected void size in the independent model

For a class with probability$p$, the probability that a fixed interval of length$L$ contains no hits from this class is

$$(1-p)^L.$$

For the class6 , the following applies

$$p_6 = 0{,}118523164.$$

Thus,

$$1-p_6 = 0{,}881476836.$$

For a 6-free segment of length$L$, the independent reference model yields

$$P_L = (0{,}881476836)^L.$$

This expression decreases exponentially with$L$.
Even at

$$L = 1.000$$

the probability of a specific, completely 6-free interval is extremely small. However, the observed voids have lengths ranging from several million to

$$34.147.153.$$

The largest observed 6-void is therefore practically incompatible with an independent random model of the measured global density.
This statement does not mean that a random event would be logically impossible. It means that the independent model does not provide a plausible description of the observed spatial distribution.

16.4 Frequency as an insufficient explanation

The lower proportion of the ‘6 ’ class explains why their average distances may be greater than those of the more frequent classes.
However, it does not explain:

Voids von mehreren Millionen Werten,

eine annähernde Verdopplungsleiter der größten Voids,

veränderte Klassenanteile vor und nach den Voids

and

eine eigene Übergangsgrammatik innerhalb der Voids.

These properties relate to the spatial order of the sequence and not merely to the number of class hits.
The void length must therefore be treated as an independent structural parameter.

17. Empirical conclusions for the classification problem

17.1 More than a global frequency distribution

The billion-test sequence initially shows a stable global ranking:

$$2 > 1 > 3 > 6.$$

No particular internal arithmetic could be deduced from this alone. Unequal frequencies can occur in many deterministic mappings.
However, further analysis reveals several additional levels:

gerichtete Übergänge,

stark konzentrierte Klassenwörter,

nicht beobachtete Vierermuster,

klassenspezifische Lauflängen,

skalierende 6-Voids

and

regional wechselnde Übergangsgrammatiken.

Together, these properties form a much stronger empirical argument for structure than the global distribution alone.

17.2 Requirements for a direct limit rule

A possible direct function

$$\Phi(N) = E(N)$$

would not only have to reproduce the four global class proportions.
It would also have to explain why the final digits of neighbouring numbers follow one another in a strongly directed sequence, why certain short patterns occur millions of times more frequently than others, and why individual patterns of four digits are completely absent from one billion values.

It would also have to account for the special role of the ‘6’ class. This includes:

ihre geringe globale Häufigkeit,

ihre lokale Bündelung,

ihre extremen Ausschlussintervalle,

die annähernde Verdopplung ihrer größten Voids

and

die Umstellung der gesamten lokalen Klassenordnung während dieser Voids.

Given this multi-scale structure, a rule based solely on a fixed, small residual class seems unlikely. A direct classification would have to capture both local and large-scale properties of the natural numbers.

17.3 Evidence of a hidden class arithmetic

The empirical findings do not yet prove the existence of a closed triatrad arithmetic. However, they provide clear indications that the four terminal classes do not occur independently or in a spatially unstructured manner.
In particular, the combination of preferred transition loops and large-scale void regimes suggests that the same class assignment operates across different scales.

Locally, it appears as a directed grammar:

$$1 \to 2 \to \begin{cases} 3, \\ 6 \end{cases} \to 1.$$

On a large scale, it appears as an alternation between normal regions, transition zones and extensive 6-exclusion regimes.
Within the exclusion regimes, the global grammar is characterised by

$$1 \to 3 \to 1$$

and

$$2 \to 3 \to 1$$

.
Any future direct class arithmetic would need to merge these two levels of description into a single rule.

17.4 Limits of the empirical statement

The main test series comprises exactly one billion consecutive initial values in the range

$$10 \leq N \leq 1.000.000.009.$$

All statistical statements initially apply to this range.
In particular, it does not yet follow from the observed doubling sequence that the same structure persists unchanged for arbitrarily large N . Nor is it proven that the 23 within the billion missing sets of four are globally impossible.

However, the data show that these structures are stable enough across a very large contiguous range to be treated as independent subjects of research.

Further numerical ranges may confirm, refine or supplement the observed rules with new regimes. The present billion-test series therefore does not constitute a definitive statistical theory, but rather an initial comprehensive empirical mapping of the triatrad sequence.

17.5 Provision of the analysis data

The analysis tables generated from the validated database are made available as a supplementary compressed dataset accompanying the Zenodo publication.
The supplementary dataset contains the machine-readable CSV analyses of the global and local class distributions, the transitions, the run lengths, the class words, the voids, and the pre-, inner and post-zones of the large 6-voids.

The underlying SQLite database is not published. The programmes used for generation, validation and analysis are fully documented in the appendix ‘Scripts used in the paper’. This ensures that the calculation logic, encoding, decoding and statistical analysis remain traceable, whilst the published supplementary data is limited to the tables relevant to the results.

18. Conclusion of the empirical study

The empirical investigation of one billion consecutive natural numbers shows that the canonical triatrad not only forms a complete terminal set of four in the recursive triangular-number descent. The four terminal values generate a distinct, multi-scale order along the natural numbers.

Globally, the four classes are permanently weighted unequally. Locally, they form directed transitions, preferred loops, class-specific repetition boundaries and, in some cases, completely excluded class words. On a large scale, the class ‘6 ’ in particular possesses exclusion intervals of up to

$$34.147.153$$

consecutive natural numbers. The largest of these voids follow a striking, approximate doubling structure in terms of position and length.

Within the 6-voids, it is not just a single terminal value that disappears. The entire local class distribution shifts to a different regime. The class3 takes on a significant portion of the missing structure; the class1 is reinforced; the class2 loses importance; and the transition loop via the class6 , which dominates outside this region, is replaced by a strong link between the classes1 and3 . This shift begins even before the actual zero zone and only gradually subsides after its end.

The terminal classes therefore occur neither independently nor with a random local distribution. Their frequencies, sequences and exclusion zones are interlinked. The empirical results do not yet prove direct triatrad arithmetic, but they do already significantly narrow down its possible form.

The billion-test series therefore lends additional weight to the central open question:

Welche intrinsische arithmetische Eigenschaft von N bestimmt unmittelbar seinen

Endwert $E(N) \in \{1,2,3,6\}$?

The theoretical triatrad theorem proves that every positive natural$N$ belongs to exactly one of the four trees. The empirical investigation further shows that these four trees do not interlock arbitrarily along the natural numbers, but form a hitherto unexplained class order with local transition rules and large-scale distribution regimes.

19. Discovery of the Triatrad in KQ

19.1 Historical background

The canonical Triatrad was not initially derived from a general investigation of the operator

$$F(N) = N - T_{t(N)}$$

. Its discovery arose from earlier investigations within KQ.
In the course of these investigations, descent paths leading from large initial values to small terminal residual values were repeatedly examined. Initially, the values

$$1, \quad 2, \quad 3$$

appeared as terminal values.
At that stage, there was as yet no general proof as to which positive terminal values were actually possible. The observed values could therefore initially only be understood as empirical terminal values.

However, the KQ structure suggested that the set visible up to that point

$$\{1,2,3\}$$

was not yet complete. Within the recursive KQ analysis, the expectation of a fourth terminal value emerged.
This fourth value was structurally defined as

$$6$$

.
This led to the prediction of the complete set of four

$$\{1,2,3,6\}$$

even before the formulated general Triatrad theorem was known.

19.2 From three observed terminal values to the set of four

The values1 ,2 and3 appeared early on as actual endpoints of calculated descent paths. However, their occurrence alone still left several possibilities open.

They could have been the first three elements of a larger terminal set. Similarly, it could not initially be ruled out that further, significantly larger terminal values might occur.

The KQ analysis, on the other hand, led to a limited structure of four. In addition to the values already observed

$$1, \quad 2, \quad 3$$

, a fourth terminal value must therefore exist.
The recursive structure led to

$$6.$$

This did not merely suggest another possible candidate. The KQ structure provided a concrete prediction:

$$\mathcal{T}_{\mathrm{KQ}} = \{1,2,3,6\}.$$

This prediction initially took the form of a structural inference within KQ. For its immediate confirmation, a specific, larger initial value was still required, the complete descent of which actually ended at6 .

19.3 The search for the fourth terminal value

Following the structural identification of ‘6 ’, the search began for a non-terminal value

$$N > 6$$

with

$$E(N) = 6.$$

This search was by no means trivial, as many small and medium-sized initial values led to the already known terminal values1 ,2 or3 .
The aim was therefore not to search retrospectively for an explanation for a randomly found terminal value. Rather, the structural prediction of6 and existed first, and a targeted search was then conducted for a complete descent path that confirmed this prediction.

The first suitable path was found for

$$N = 82$$

found.
The following holds

$$t(82) = 9$$

and

$$T_9 = \frac{9 \cdot 10}{2} = 45.$$

It follows that

$$82 \to 37.$$

For the next value, we have

$$t(37) = 6$$

and

$$T_6 = \frac{6 \cdot 7}{2} = 21.$$

Thus, we obtain

$$37 \to 16.$$

Subsequently, the following holds

$$t(16) = 4$$

and

$$T_4 = \frac{4 \cdot 5}{2} = 10.$$

Therefore, it follows that

$$16 \to 6.$$

For6 ,

$$t(6) = 3$$

and

$$T_3 = 6.$$

Thus, the next formal value would be

$$F(6) = 0.$$

The complete positive descent path is therefore

$$82 \to 37 \to 16 \to 6.$$

This confirms, for the first time, the structurally expected fourth terminal class by means of a specific non-terminal initial value:

$$E(82) = 6.$$

19.4 Backtracking representation of the path found

The path found can also be read as part of the back-tracking tree of the `$\mathcal{C}_6$` class:

$$6 \leftarrow 16 \leftarrow 37 \leftarrow 82.$$

Each arrow indicates that, on the next descent, the value on the right will drop to the value on the left.

For the first step, the following applies

$$16 = 6 + T_4.$$

For the second step, the following applies

$$37 = 16 + T_6.$$

For the third step, the following applies

$$82 = 37 + T_9.$$

The path is thus also a specific finite predecessor chain of the root6 .

This representation links the early KQ discovery directly to the Vierbaum decomposition developed later in Chapter 7. The value82 lies in the back-tree of the root6 and therefore belongs to the class

$$\mathcal{C}_6.$$

19.5 Relationship between the KQ prediction and the Triatrad theorem

The KQ prediction and the Triatrad Theorem fulfil different scientific functions.

The KQ investigation led to the discovery and structural prediction of the set of four

$$\{1,2,3,6\}.$$

The later Triatrad theorem, by contrast, proves—independently of this history of discovery—that no further positive terminal values can exist.

The two approaches can therefore be distinguished as follows:

KQ → strukturelle Vorhersage,

Triatrad-Satz → allgemeiner Vollständigkeitsbeweis.

KQ predicted the fourth limit before a complete general proof for the set of limits was available. The Triatrad theorem subsequently provides a classical explanation as to why

$$1, \quad 2, \quad 3, \quad 6$$

and not a fifth positive value can be terminal.

The KQ derivation itself is not the subject of this paper. For the present work, it is merely crucial that KQ served as a space for discovery and made the complete set of four visible even before its general arithmetic justification.

19.6 KQ and KFS as a geometric space of discovery

Within KQ and KFS, a value N is not regarded solely as an isolated point in the set of natural numbers. It possesses a group-specific position, an adaptive motor and associated shell partners.

For the descent operator used here, the motor is

$$K(N) = T_{t(N)}.$$

The KFS representation thus takes the form

$$N - T_{t(N)} \leftrightarrow N \leftrightarrow N + T_{t(N)}.$$

The triatrad descent follows the left-hand shell partner in each case:

$$N \to N - T_{t(N)}.$$

After each step, the motor is redetermined from the new value. The descent is therefore adaptive. The same triangular number is not subtracted repeatedly.

This geometric representation revealed the recursive structure of the terminal values before their complete classical characterisation was available.

This KQ and KFS language is not required for the general proof of the Triatrad Theorem. However, it explains the research path by which the set of four was first recognised.

19.7 No second line of proof

This chapter is not intended to provide an alternative complete derivation of the Triatrad from KQ.

Such a derivation would require extensive additional definitions and shift the focus of this work. The main mathematical theorem has already been proven independently.

The aim of this chapter is therefore limited:

Firstly, it documents that the terminal values1 ,2 and 3 initially became apparent empirically in KQ descents.

Secondly, it is noted that KQ suggested a fourth limit value.

Thirdly, the recursive determination of this limit value as6 is described.

Fourthly,

$$82 \to 37 \to 16 \to 6$$

the first concrete, confirmatory descent path is specified.
This leaves the scientific division of roles unambiguous:

$$\text{KQ und KFS} = \text{Entdeckungszusammenhang,}$$

$$\text{Triatrad-Satz} = \text{allgemeiner Begründungszusammenhang.}$$

19.8 Observed descent times for large values

The early KQ investigations did not only cover small examples. Very large initial values with bit lengths of approximately2048 and4096 were also investigated.
This revealed a striking correlation between the actual bit length of the initial value and the number of complete descent steps.

For tested values with

$$\text{bitlen}(N) = 2048$$

There were approximately

$$2046$$

complete descents were observed.
For verified values with

$$\text{bitlen}(N) = 4096$$

the observed decline figures were approximately within the range

$$4093 \text{ bis } 4096.$$

These observations lead to the scaling relation

$$L(N) \approx \mathrm{bitlen}(N),$$

where$L(N)$ denotes the number of complete descent transitions up to the terminal value.

This statement should be regarded as an empirical hypothesis for the time being. The available large test values show a clear correlation between bit length and descent time, but are not sufficient to establish a general theorem covering all positive natural numbers.

19.9 Theoretical explanation of the order of magnitude

The observed correlation between bit length and descent time can be plausibly explained by the form of the operator.

For large$N$, the following holds

$$t(N) \approx \sqrt{N}.$$

The corresponding triangular number is therefore approximately of the order of

$$T_{t(N)} = \frac{t(N)(t(N)+1)}{2} \approx \frac{N}{2}.$$

It follows that

$$F(N) = N - T_{t(N)} \approx \frac{N}{2}.$$

Halving a positive integer typically reduces its binary length by approximately one digit. Thus, approximately

$$\mathrm{bitlen}(F(N)) \approx \mathrm{bitlen}(N) - 1.$$

If this process is continued recursively, it is to be expected that a value of$b$ bits requires approximately$b$ descent steps to reach the small terminal region.

This leads to the hypothesis

$$L(N) \sim \log_2 N$$

or

$$L(N) \approx \mathrm{bitlen}(N).$$

19.10 Exact deviation from halving

Halving is only the dominant term. The operator has an exact correction form within an interval.

Let

$$m = t(N)$$

and

$$N = m(m-1) + r$$

where

$$0 \le r \le 2m - 1.$$

Then the following holds

$$F(N) = \frac{m(m-3)}{2} + r.$$

Since

$$\frac{N}{2} = \frac{m(m-1) + r}{2},$$

it follows

$$F(N) - \frac{N}{2} = \frac{r}{2} - m.$$

Thus, we obtain the exact expression

$$F(N) = \frac{N}{2} + \frac{r}{2} - m.$$

The term

$$\delta(N) = \frac{r}{2} - m$$

describes the deviation of a single step from the ideal halving.

Since

$$0 \le r \le 2m - 1$$

the following holds

$$-m \le \delta(N) \le -\frac{1}{2}.$$

It follows, therefore, that for every positive natural number

$$F(N) < \frac{N}{2}.$$

Each step therefore always results in a greater reduction than an exact halving. The extent of this additional reduction depends on the position r within the interval. The total path length is influenced by the sequence of these correction terms.

19.11 Hypothesis of approximate bit loss

The empirical large-number tests and the asymptotic halving form lead to the following hypothesis.

Hypothesis 1. For large positive natural starting values, a complete descent step reduces the binary length of the current value by approximately one bit on average. Therefore, the number of descent steps to the triatrad final value is of the order of magnitude of the bit length of the starting value:

$$L(N) \approx \mathrm{bitlen}(N).$$

This formulation refers to an approximate loss of information. It does not claim that exactly one bit is lost in every single step.

The exact change depends on

$$\delta_k = \frac{r_k}{2} - m_k$$

, whereby for the$k$ th descent value

$$N_k = m_k(m_k - 1) + r_k$$

.

The complete path can thus be written as

$$N_{k+1} = \frac{N_k}{2} + \delta_k.$$

Because

$$0 \le r_k \le 2m_k - 1$$

the following always holds

$$\delta_k \le -\frac{1}{2}.$$

It follows, therefore, for every descent transition

$$N_{k+1} < \frac{N_k}{2}.$$

Halving determines the coarse envelope. The sequence

$$\delta_0, \delta_1, \ldots, \delta_{L(N)-1}$$

determines the finer deviations and possibly also the terminal class assignment.

19.12 Information reduction

The approximate halving allows for an information-theoretic interpretation of the descent.

An initial value with$b$ binary digits has a magnitude of the order

$$b \approx log_2 N.$$

For every descent transition actually carried out, the following applies

$$N_{k+1} < \frac{N_k}{2}.$$

It follows that

$$bitlen\left(N_{(k+1)}\right) \leq bitlen(N_k) - 1$$

Each descent transition thus reduces the binary length required to represent the current value by at least one digit. In some steps, several binary digits may be lost.

The descent can therefore be described as a deterministic reduction in information:

$$N_0 \to N_1 \to N_2 \to \cdots \to E(N_0).$$

In the end, one of the four terminal values remains

$$E(N_0) \in 1,2,3,6.$$

The initial value thus gradually loses almost all of its magnitude information. However, it retains terminal class information that uniquely assigns it to one of the four triatrad trees.

This final information is not determined by bit length alone. Two very large values may have the same or almost the same bit length and yet end up in different Triatrad classes.

The coarse magnitude information is therefore reduced through the ongoing magnitude reduction, whilst the terminal class information is determined by the complete sequence of discrete corrections

19.13 Open questions regarding descent duration

The hypothesis put forward so far requires a much broader and more systematic investigation.

Since each descent transition reduces the binary length by at least one digit, the upper bound for the number$L(N)$ of actual descent transitions is

$$L(N) \leq bitlen(N) - bitlen\big(E(N)\big).$$

A first precise research question therefore concerns the difference from the maximum possible path length. For this purpose, the non-negative quantity

$$D(N) = bitlen(N) - bitlen\big(E(N)\big) - L(N)$$

be defined.

It remains an open question as to which bounds apply to$D(N)$ and whether a global constant$C$ exists such that

$$0 \leq D(N) \leq C$$

N holds for all sufficiently large .

A weaker asymptotic question is whether

$$L(N) = bitlen(N) + o\big(bitlen(N)\big)$$

holds. Since the terminal value has only a finite bit length, it must also be investigated whether

$$\frac{L(N)}{bitlen(N)} \to 1$$

holds for$N \to \infty$ in a suitable sense.

Further open questions concern the terminal classes:

Further open questions concern the terminal classes. It remains to be investigated whether the classesC_1 ,C_2 ,C_3 andC_6 have different path-length profiles. It is also an open question whether the distance$D(N)$ depends systematically on the terminal value$E(N)$, and what role the correction sequence$\delta_k = \frac{r_k}{2} - m_k$ plays in the path length and the terminal class assignment. Furthermore, it must be examined whether there are descent paths that are exceptionally short relative to the bit length, and how frequently descent paths lie close to their theoretical maximum length.

These questions are not conclusively investigated in the present work. They constitute a separate area of research for future work

19.14 Significance of the hypothesis for Triatrad research

The hypothesis of approximate bit loss explains the length of the path, not its terminal value.

For large$N$, all four terminal classes share the same rough exponential envelope:

$$N_k \approx \frac{N_0}{2^k}.$$

From this envelope curve alone, it is not possible to determine whether the path, for

$$1, \quad 2, \quad 3$$

or

$$6$$

.

Rather, the class information lies in the discrete deviations

$$\delta_k.$$

This results in two distinct levels:

Halbierungsanteil → Weglänge,

Korrekturfolge → mögliche Klasseninformation.

This distinction illustrates why the descent time can be described relatively simply by the bit length, whilst the direct determination of the final value remains an open question.

The hypothesis thus reinforces the central classification problem. It shows that the operator reduces the magnitude information of an arbitrarily large$N$ to a number of steps approximately proportional to the bit length, whilst preserving non-trivial four-valued terminal information.

20. Further structural questions

20.1 Direct determination of the final value

The key unresolved problem remains the direct determination of

$$E(N)$$

without carrying out the full descent.

We are seeking an arithmetic mapping

$$\Phi: \mathbb{N}_{>0} \to \{1,2,3,6\}$$

such that

$$\Phi(N) = E(N)$$

for all positive natural numbers.

Such a mapping would have to account for both the local transition grammar and the large-scale void regimes.

20.2 Structure of the correction sequence

The exact recursion

$$N_{k+1} = \frac{N_k}{2} + \delta_k$$

with

$$\delta_k = \frac{r_k}{2} - m_k$$

suggests that the sequence

$$\delta_0, \delta_1, \dots, \delta_{L(N)-1}$$

as a separate object.

Since

$$0 \leq r_k \leq 2m_k - 1$$

the following holds for every descent transition

$$-m_k \leq \delta_k \leq -\frac{1}{2}.$$

The correction terms are therefore always negative. It is therefore not necessary to distinguish between positive and negative deviations.

Possible metrics are, instead, the cumulative correction

$$K(N) = \sum_{k=0}^{L(N)-1} \delta_k,$$

the mean correction

$$\overline{\delta}(N) = \frac{1}{L(N)} \sum_{k=0}^{L(N)-1} \delta_k$$

and the number of particularly large deviations from the halving. For a given bound$\lambda > 0$, for example,

$$S_\lambda(N) = \#\{k : \delta_k \leq -\lambda\}$$

be defined.

It remains to be examined whether different triatrad classes possess characteristic correction profiles

20.3 Symbolic path encoding

Each descent path generates a sequence of interval indices

$$m_0, m_1, \ldots, m_{L(N)-1}$$

and a sequence of positions

$$r_0, r_1, \ldots, r_{L(N)-1}.$$

This data can be understood as a symbolic encoding of the path.

It remains an open question whether a condensed signature of these sequences determines the final value. A direct class rule could possibly be derived from a few aggregated properties of the path profile, without explicitly reconstructing the complete descent.

20.4 Backward trees as classification objects

The four classes have complete inverse generation trees.

For a node$M$, its predecessors are derived from

$$N = M + T_m$$

subject to the condition

$$\frac{m(m-3)}{2} \leq M \leq \frac{m(m+1)}{2} - 1.$$

A possible direct characterisation could therefore arise not from the forward descent, but from structural properties of the four backward trees.
Among other things, the following should be investigated:

Verzweigungsgrade,

zulässige Indexfolgen,

fehlende Vorgängermuster,

Dichteentwicklung der Baumebenen

and

symbolische Signaturen der vier Wurzeln.

20.5 Explanation of the 6-voids

The large 6-voids represent one of the most prominent open structural problems.
The aim is to find a direct arithmetic description of the intervals

$$[a_k, b_k],$$

in which

$$E(N) \neq 6$$

for all

$$a_k \leq N \leq b_k$$

holds.
In particular, it remains to be clarified why the largest observed void lengths and their starting points exhibit an approximate doubling structure.

A complete theory would have to explain not only the zero zone, but also the observed pre- and post-zones, as well as the transition between different local grammars.

20.6 Global distribution laws

The measured class proportions

$$P_1 \approx 29{,}34\,\%,$$

$$P_2 \approx 39{,}39\,\%,$$

$$P_3 \approx 19{,}41\,\%,$$

$$P_6 \approx 11{,}85\,\%$$

raise the question of possible asymptotic densities.

It remains an open question whether the limit values

$$\lim_{X\to\infty} \frac{C_e(X)}{X}$$

exist for all

$$e \in \{1,2,3,6\}$$

exist.

It is also unknown whether the ranking observed within the first billion

$$2 > 1 > 3 > 6$$

is preserved globally.

20.7 Prohibited patterns

Within the billion-test series, no four-pattern sequences of the form23 were observed. It remains to be examined whether these patterns are merely absent from the range investigated or whether some of them are generally impossible.

A formal exclusion of a four-element pattern

$$(a, b, c, d)$$

would imply that no positive natural number$N$ exists such that

$$E(N) = a,$$

$$E(N+1) = b,$$

$$E(N+2) = c,$$

$$E(N+3) = d.$$

Such a theorem would constitute a direct local restriction on the limit function and could provide important clues as to its unknown arithmetic form.

20.8 Future research

This work resolves the terminal completeness problem and documents the first large-scale empirical structure of the four classes.

It does not claim to have already answered all the questions arising from this.

In particular, the following topics remain to be addressed in future work:

exakte Theorie der Abstiegsdauer,

direkte Klassenarithmetik,
asymptotische Klassendichten,
Theorie der 6-Voids,
formale Musterverbote

and

vollständige Struktur der vier Rückwärtsbäume.

The hypotheses set out in this paper are intended to facilitate these investigations in a targeted manner and not to pre-empt them through premature generalisations.

21. Concluding Remarks

21.1 Established mathematical results

This paper investigates the recursive descent of triangular numbers

$$F(N) = N - T_{t(N)},$$

with

$$t(N) = \left\lfloor \frac{\sqrt{4N+1}+1}{2} \right\rfloor$$

and

$$T_m = \frac{m(m+1)}{2}.$$

For every positive non-terminal initial value, the following holds

$$1 \leq F(N) < N.$$

Every descent path is therefore strictly decreasing and reaches a positive terminal value after a finite number of steps. Since the operator is deterministic, every

$$N \in \mathbb{N}_{>0}$$

has a uniquely determined terminal value.

A complete analysis of the terminal condition shows that only

$$1, \quad 2, \quad 3, \quad 6$$

are possible as positive terminal values. Thus, the canonical triatrad is defined as

$$\mathcal{T} = \{1,2,3,6\}.$$

For the terminal value function

$$E: \mathbb{N}_{>0} \to \mathcal{T}$$

the following therefore holds for every positive natural number N

$$E(N) \in \{1,2,3,6\}.$$

A fifth positive terminal value is ruled out.

From the uniqueness of the limit function, the following four classes follow

$$\mathcal{C}_e = \{N \in \mathbb{N}_{>0} : E(N) = e\}, \qquad e \in \{1,2,3,6\}.$$

These classes form a disjoint and complete decomposition:

$$\mathbb{N}_{>0} = \mathcal{C}_1 \,\dot\cup\, \mathcal{C}_2 \,\dot\cup\, \mathcal{C}_3 \,\dot\cup\, \mathcal{C}_6.$$

In the directed graph representation, every non-terminal value has exactly one outgoing descending path. The four terminal values form the roots of the corresponding inverse trees. This results in a canonical four-tree decomposition of the positive natural numbers.
The fundamental mathematical statement of this work is therefore:

Jede positive natürliche Zahl gehört eindeutig genau einem der vier Triatrad-Bäume an.

21.2 Recursion and direct classification

The recursive determination of terminal values is fully defined:

$$E(N) = E(F(N))$$

for every non-terminal value N , together with the initial values

$$E(1) = 1, \qquad E(2) = 2, \qquad E(3) = 3, \qquad E(6) = 6.$$

Thus, the class of every positive natural number can be determined by repeated application of the operator.
However, this recursive determinability is not equivalent to a direct arithmetic characterisation. At present, no closed-form rule is known which, based solely on the initial value N and without executing the descent, allows one to immediately determine which of the four terminal values will be reached.

The central classification problem therefore remains open:

Welche intrinsische arithmetische Eigenschaft von N bestimmt $E(N)$?

We are seeking a direct mapping

$$\Phi: \mathbb{N}_{>0} \to \{1,2,3,6\}$$

with

$$\Phi(N) = E(N)$$

for all positive natural numbers.
The Triatrad Theorem provides a complete answer as to which terminal values are possible. It does not yet explain why a particular$N$ belongs to a specific one of the four trees.

21.3 Results of the main numerical investigation

The limit function was evaluated for exactly one billion consecutive natural numbers in the range

$$10 \leq N \leq 1.000.000.009$$

calculated.

The full distribution was as follows:

$$E(N) = 1: 293.421.555,$$
$$E(N) = 2: 393.924.939,$$
$$E(N) = 3: 194.130.342,$$
$$E(N) = 6: 118.523.164.$$

This corresponds to the shares

$$P_1 = 29{,}3421555\,\%,$$
$$P_2 = 39{,}3924939\,\%,$$
$$P_3 = 19{,}4130342\,\%,$$
$$P_6 = 11{,}8523164\,\%.$$

The overall ranking is therefore

$$2 > 1 > 3 > 6.$$

The four classes do not occur with equal frequency. At the same time, the distribution is roughly in line with the ratio

$$3: 4: 2: 1,$$

but shows a noticeable shift in favour of the class 6 compared with this simple reference.

The global distribution is only the first empirical level. It describes the total weights of the four trees, not their arrangement along the natural numbers.

21.4 Local order of the sequence of terminal values

An examination of neighbouring terminal values showed that the classes do not occur independently of one another.

The sequence

$$E(10), E(11), \ldots, E(1.000.000.009)$$

exhibits directed transitions, highly unequal pattern frequencies and class-specific repetition limits.

Local movements of the form

$$1 \to 2 \to 3 \to 1$$

and

$$1 \to 2 \to 6 \to 1.$$

The class2 frequently acts as a branching point in this context. From there, the local sequence tends to lead to3 or6 . Both branches subsequently often return to the class1 .

This simplified grammar is not a rule without exceptions. However, it describes the statistically dominant local movement.

Among the64 possible triplet patterns, all combinations occur. Their frequencies, however, differ by a ratio of several million to one. The most common triplet pattern

$$2 \to 3 \to 1$$

has been observed more than107 million times, whilst

$$3 \to 6 \to 3$$

it occurred only24 times.

In the case of four-word patterns, completely missing sequences appear for the first time. Of the

$$4^4 = 256$$

theoretically possible four-word sequences, not a single one was

$$23$$

.

The sequence of final values therefore has a significantly restricted local transition structure.

21.5 The special status of classes 3 and 6

Classes3 and 6 exhibit a remarkable relationship.

In large-scale distributions, they appear to partially substitute for one another. In particular, the proportion of class3 rises sharply in large 6-free regions.

Locally, however, the two classes are almost never found immediately adjacent to one another. Patterns involving close alternations between3 and 6 are among the rarest triplets in the entire study.

The empirical relationship can therefore be summarised as

großräumige Kopplung bei lokaler Trennung.

This suggests that both classes may fulfil alternative roles within the same overarching structure, without frequently transitioning directly into one another.

21.6 The 6-voids as the most significant large-scale finding

The most striking empirical finding concerns the extensive intervals in which the ‘6 ’ class is completely absent.
The largest 6-void observed within the first billion begins at

$$783.765.283$$

and has a length of

$$34.147.153.$$

For more than34 million consecutive natural numbers, the following holds there

$$E(N) \neq 6.$$

For the ten largest observed 6-voids, there is a striking approximate doubling pattern in both their lengths and their positions.
The void lengths in this sequence range from

$$69.461$$

to

$$34.147.153.$$

Approximatively, the following holds over several stages

$$V_{k+1} \approx 2V_k.$$

An independent random arrangement with the same global 6-frequency would not plausibly explain voids of this order of magnitude. The random comparison serves solely as a reference model, as the triatrad sequence itself is entirely deterministic.
The 6-voids demonstrate that the ‘6 ’ class does not merely occur less frequently on a global scale. It possesses its own large-scale geometry.

21.7 Voids as independent distribution regimes

Within the large 6-voids, it is not only the class6 that disappears. The entire local distribution changes.
Outside the voids, the class ‘2 ’ accounts for nearly$40\,\%$, whilst the class ‘3 ’ stands at approximately$18{,}5\,\%$.

Within the Voids, the distribution shifts approximately to

$$P_1 \approx 37{,}24\,\%,$$
$$P_2 \approx 31{,}99\,\%,$$
$$P_3 \approx 30{,}77\,\%,$$
$$P_6 = 0.$$

The missing class ‘6 ’ is not simply distributed proportionally amongst the three remaining classes. In particular, the class ‘3 ’ gains significantly, whilst the class ‘2 ’ loses market share despite the loss of ‘6 ’.
The transition can be described approximately as follows:

$$6 \downarrow 0, \qquad 3 \uparrow\uparrow, \qquad 1 \uparrow, \qquad 2 \downarrow.$$

The regime change begins even before the actual void and only subsides gradually after it has ended.
Thus, the large 6-Voids have a pre-zone, a complete zero-zone and a post-zone.

21.8 Changes to the transitional grammar within the voids

The transitions between neighbouring end values also change within the voids.
Outside the voids,

$$2 \to 6 \to 1$$

one of the two dominant branches.
Within a 6-void, this path is completely ruled out. Instead, there is an increased tendency for

$$2 \to 3 \to 1$$

and

$$1 \to 3 \to 1.$$

This connection becomes particularly clear

$$1 \leftrightarrow 3.$$

Within the Voids, a value of the class3 is followed with a conditional probability of approximately

$$98{,}39\,\%$$

a value from the class ‘1 ’.

The void structure is thus not merely a local shift in frequency; it possesses its own transition grammar.

21.9 Significance of the empirical results

The empirical findings do not yet prove direct class arithmetic. However, they show that the sequence of final values cannot be understood as an unstructured four-valued mapping.

The following were observed:

ungleiche globale Klassenanteile,

gerichtete Nachbarschaftsbeziehungen,

stark konzentrierte Klassenwörter,

vollständig fehlende Vierermuster,

klassenspezifische Lauflängen,

skalierende 6-Voids

and

regional wechselnde Übergangsgrammatiken.

These properties must be jointly explained by any future direct theory of the end-value function.

A possible class rule must therefore not only reproduce the global frequency; it must also account for local transitions, pattern prohibitions, void boundaries and regime changes.

21.10 History of discovery in KQ

The Triatrad first became apparent within KQ.

Early descent studies revealed the terminal values

$$1, \quad 2, \quad 3.$$

However, the KQ structure suggested the existence of a fourth terminal class. This was determined recursively as

$$6$$

before a general classical completeness proof was available.

The subsequent targeted search for a non-terminal initial value with final value6 ultimately led to

$$82 \to 37 \to 16 \to 6.$$

This confirmed the structural prediction through a concrete descent path.

KQ thus served as a space for discovery. The Triatrad theorem, developed later, on the other hand, provides the independent general justification for the fact that only

$$\{1,2,3,6\}$$

are possible as positive terminal values.

21.11 Hypothesis regarding the duration of descent

Investigations of very large initial values led to a further hypothesis.

For tested values with an actual bit length of2048 , approximately

$$2046$$

complete descent stages were observed.

For tested values with a bit length of4096 , the descent counts were approximately

$$4093 \text{ bis } 4096.$$

This yields the empirical approximation

$$L(N) \approx \mathrm{bitlen}(N).$$

This observation is consistent with the asymptotic form of the operator. For large$N$, the following holds

$$F(N) \approx \frac{N}{2}.$$

A halving reduces the binary length by an average of approximately one bit. The descent can therefore be understood as a step-by-step reduction in the amount of information.

This statement is explicitly formulated as a hypothesis. In particular, it remains an open question as to which exact bounds apply to

$$L(N) - \mathrm{bitlen}(N)$$

and whether the deviation is globally bounded.

21.12 Information content and terminal class information

The descent time and the terminal value class describe two different properties of the path.

The asymptotic halving determines the rough trend in size:

$$N_k \approx \frac{N_0}{2^k}.$$

It explains why the path length roughly follows the bit length. However, it does not explain which of the four terminal values is reached.

The terminal class information lies in the discrete deviations from the ideal halving. If we write

$$N_k = m_k(m_k - 1) + r_k,$$

then the following holds

$$N_{k+1} = \frac{N_k}{2} + \delta_k$$

where

$$\delta_k = \frac{r_k}{2} - m_k.$$

Since

$$0 \le r_k \le 2m_k - 1$$

the following always holds

$$\delta_k \le -\frac{1}{2}.$$

Each descent transition therefore results in a greater reduction than an exact halving. The exact magnitude of this additional reduction depends on the respective interval positionr_k .

The rough magnitude information is gradually reduced during the descent. At the end, however, a four-valued piece of information remains:

$$E(N) \in 1,2,3,6.$$

This terminal information uniquely assigns the original value to one of the four trees.

21.13 Key outstanding issues

Several clearly defined research questions arise from this work.
The most important remains direct classification:

Existiert eine geschlossene arithmetische Regel für $E(N)$?

In addition, the following questions in particular remain open:

Existieren asymptotische Dichten der vier Klassen?

Wie lassen sich die großen 6-Voids direkt beschreiben?
Setzt sich ihre Verdopplungsstruktur für größere N fort?
Sind die nicht beobachteten Vierermuster allgemein verboten?
Besitzen die vier Rückwärtsbäume charakteristische Signaturen?
Welche exakten Schranken gelten für die Abstiegsdauer?
Bestimmt die Folge der Halbierungskorrekturen den Endwert direkt?

These questions are not answered by the Triatrad theorem. However, the theorem provides, for the first time, the firm mathematical framework within which they can be precisely formulated.

21.14 Limitations of the present study

The theoretical statements regarding finiteness, uniqueness and the complete terminal set apply to all positive natural numbers.
The empirical results on distribution, transition and voids, on the other hand, relate to the range under investigation

$$10 \leq N \leq 1.000.000.009.$$

It does not automatically follow from the billion-test series that all observed frequencies, patterns and scalings continue unchanged into infinity.
In particular, the measured class proportions are not proven limit densities. The doubling structure of the 6-voids is a strong empirical finding, but not yet a general law. Similarly, the absence of certain four-patterns within the billion does not yet constitute a global proof of impossibility.

The study provides an initial large-scale mapping of the Triatrad. It forms a starting point for further evidence and larger test series.

21.15 Scientific Classification

The canonical triatrad possesses three distinct levels.
Firstly, it is a proven terminal set:

$$\mathcal{T} = \{1,2,3,6\}.$$

Secondly, it generates a complete partition of the positive natural numbers:

$$\mathbb{N}_{>0} = \mathcal{C}_1 \dot{\cup} \mathcal{C}_2 \dot{\cup} \mathcal{C}_3 \dot{\cup} \mathcal{C}_6.$$

Thirdly, it presents an open problem in arithmetic classification, as the direct cause of the mapping

$$N \mapsto E(N)$$

is still unknown.
The theoretical level proves the existence of the four classes. The empirical level shows that these classes possess a complex local and large-scale order. The open arithmetic level seeks to uncover the hidden mechanism behind this order.

21.16 Concluding assessment

This paper demonstrates that the recursive descent of triangular numbers does not terminate in an arbitrary set of small residual values. All positive natural numbers are reduced to exactly four terminal values.
These four values do not merely constitute a list of exceptions. They are the roots of a complete four-tree decomposition of the positive natural numbers.

Furthermore, the billion-test series shows that the four trees do not interlock arbitrarily along the number line. Their members form directed transitions, heavily weighted class words, locally excluded patterns and large-scale distribution regimes.

In particular, the class ‘6 ’ possesses an extraordinary spatial structure. Its extensive voids and the associated changes in the remaining classes point to a hitherto unknown deterministic ordering mechanism.

The central task of future research therefore no longer lies in finding the possible final values. These are fully determined.

The remaining task is:

> Die direkte arithmetische Ursache der Vierbaumzuordnung zu bestimmen.

The canonical triatrad is thus simultaneously a proven terminal system, a complete Vierbaum decomposition of the positive natural numbers, and the starting point of an open arithmetic classification problem.

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