The 6 voids of the canonical triad

Arithmetic derivation of macroscopic gaps in the backward tree C_6

Version 1:
Publication date: 20 August 2026

Achim Meißner

Email: service@ameis-web.de

1 The macroscopic 6-voids

1.1 Current status

In the paper *'The canonical triatrad of natural numbers'*[1] , a complete decomposition of the positive natural numbers into four disjoint backward trees with terminal values$1,2,3,6$ was described. As part of the large-scale numerical investigation carried out there, the class with terminal value6 stood out in particular due to extensive contiguous intervals in which not a single value from this class occurs. These intervals were termed 6-voids. Among other things, the original investigation revealed a 6-void containing more than34 million consecutive natural numbers. Its structural cause remained unresolved at that stage.

The 'backwards analysis' of the 6-tree, which has since been developed, provides a complete explanation of these macroscopic voids. It shows that these do not arise as random gaps within the class, but as spaces between successive depth layers of the backwards tree. In this process, the boundaries of the observed voids are determined solely from the backwards structure.

Further empirical testing has since revealed the immediately following macro-void

$$1\,567\,587\,904 \leq N \leq 1\,635\,831\,670$$

with a length of

$$68\,243\,767.$$

This region had previously been calculated from the backward structure. The subsequent independent forward measurement confirmed both the start and end, as well as the length, of the predicted void exactly. The last value in the classC_6 before the void is$1\,567\,587\,903$, and the first value after it is$1\,635\,831\,671$.

1.2 Derivation and proof of the 6 voids

For a terminal value$c \in \{1,2,3,6\}$, let

$$D_c(k)$$

[1] Meißner, A. (26 July 20226). The canonical triad of natural numbers. A Vierbaum decomposition and the open problem of terminal class assignment. Zenodo. https://doi.org/10.5281/zenodo.21603457

the set of all natural numbers that reach the terminal value$c$ after exactly$k$ descent steps. For the back-treeC_6 , the two outer bounds of such a depth layer are given by

$$L_k = \min D_6(k)$$

and

$$U_k = \max D_6(k)$$

.

For any node$X$ of the backward tree, its predecessors take the form

$$N = X + T_m,$$

where$T_m = m(m+1)/2$ denotes the$m$ -th triangular number. An index$m$ is valid if and only if

$$\frac{m(m-3)}{2} < X \leq \frac{m(m+1)}{2}.$$

XIf lies between two consecutive triangular numbers,

$$T_{r-1} \leq X < T_r,$$

then$X$ has exactly two regular predecessors with indices$r$ and$r+1$. In the special case

$$X = T_r - 1$$

, a third predecessor with index$r+2$ also appears. No other immediate predecessors exist.

Since both the smallest and the largest predecessors increase monotonically with the root node, it is not necessary to generate all the internal nodes of the tree in order to determine the outer boundaries of a new depth layer. It suffices to perform the recursion

$$L_{k+1} = P_-(L_k),$$

$$U_{k+1} = P_+(U_k),$$

whereP_- andP_+ denote the smallest and largest permissible predecessors, respectively.

If the upper boundary of a depth layer lies strictly before the lower boundary of the following depth layer,

$$U_k < L_{k+1},$$

then no element of the 6-tree can lie between these two boundaries. The interval

$$V_k = [U_k + 1, L_{k+1} - 1]$$

is therefore completely free of numbers with terminal value6 . Its length is

$$|V_k| = L_{k+1} - U_k - 1.$$

The macroscopic 6-voids are thus directly the gaps between successive depth layers of the back-tree.

If we set

$$H_k = L_{k+1} - U_k,$$

Thus, from the growth of the two inverse boundary maps, the following estimate follows

$$H_{k+1} \geq 2H_k - 2\sqrt{2U_k} - 5.$$

For the eleventh depth layer under consideration, the following already holds

$$L_{11} = 24\,761, \qquad U_{11} = 47\,037, \qquad L_{12} = 49\,737$$

and thus

$$H_{11} = 2700 > 12\sqrt{47\,037}.$$

Assuming

$$H_k \geq 12\sqrt{U_k}$$

it follows that

$$H_{k+1} \geq 24\sqrt{U_k} - 2\sqrt{2}\,\sqrt{U_k} - 5.$$

At the same time, the upper growth estimate holds

$$U_{k+1} \leq 3U_k.$$

Since

$$24 - 2\sqrt{2} - 12\sqrt{3} > 0$$

and the remaining positive difference for$U_k \geq 47\,037$ already exceeds the additive constant5 , it follows that

$$H_k \geq 12\sqrt{U_k} \quad \Rightarrow \quad H_{k+1} \geq 12\sqrt{U_{k+1}}.$$

By complete induction, this inequality holds for all$k \geq 11$. SinceU_k grows unbounded, it follows that

$$H_k \to \infty$$

and consequently

$$|V_k| = H_k - 1 \to \infty.$$

Thus, there are infinitely many macroscopic 6-voids, and their lengths are unbounded. The terminal values$1,2,3$ exhibit different behaviour under the same backward analysis. In their backward trees, the span ranges of successive depth layers begin to overlap after a finite number of steps. ForC_1 , this occurs from depth3 ; forC_2 , from depth1 ; and forC_3 , from depth4 . Due to the monotonicity of the inverse boundary mappings, once an overlap has occurred, it persists. The mechanism of continuous macroscopic gaps, which is effective forC_6 , cannot therefore occur in these three terminal trees.

1.3 Agreement between derivation and empirical results

The macro-voids calculated from the outer boundaries of the backward tree correspond exactly to the 6-voids found independently via forward iteration. Twelve consecutive macro-voids were confirmed for the first billion natural numbers. The next void, subsequently determined from the backward structure, was also confirmed to the exact position by the analysis of the second billion.

No.	empirical range	Empirical length	Range calculated from the proof	calculated length
1	380,543 to 398,583	18,041	380,543 to 398,583	18,041
2	762,044 to 797,754	35,711	762,044 to 797,754	35,711
3	1,526,510 to 1,595,970	69,461	1,526,510 to 1,595,970	69,461
4	3,055,136 to 3,193,548	138,413	3,055,136 to 3,193,548	138,413
5	6,114,237 to 6,387,676	273,440	6,114,237 to 6,387,676	273,440
6	12,233,988 to 12,776,201	542,214	12,233,988 to 12,776,201	542,214
7	24,477,814 to 25,555,241	1,077,428	24,477,814 to 25,555,241	1,077,428
8	48,967,315 to 51,112,916	2,145,602	48,967,315 to 51,112,916	2,145,602
9	97,947,568 to 102,234,132	4,286,565	97,947,568 to 102,234,132	4,286,565

No.	empirical range	Empirical length	Range calculated from the proof	calculated length
10	195,912,571 to 204,471,982	8,559,412	195,912,571 to 204,471,982	8,559,412
11	391,863,277 to 408,946,735	17,083,459	391,863,277 to 408,946,735	17,083,459
12	783,765,283 to 817,912,435	34,147,153	783,765,283 to 817,912,435	34,147,153
13	1,567,587,904 to 1,635,831,670	68,243,767	1,567,587,904 to 1,635,831,670	68,243,767

For all thirteen macro-voids under consideration, the following therefore holds with exact positional accuracy

$$V_k^{\mathrm{emp}} = V_k^{\mathrm{theor}}.$$

The agreement relates to the start, end and length of the interval in each case. In particular, the thirteenth void was not used to determine the backward structure, but was only confirmed by the independent forward measurement after its theoretical calculation.

1.4 Concluding remark

The macroscopic 6-voids empirically observed in the original Triatrad paper are thus explained arithmetically. They arise as necessary gaps between successive depth layers of the backward tree C_6 . There are an infinite number of such voids, and their lengths are unlimited.

The independent empirical measurement precisely confirms the intervals determined from the backward structure for thirteen consecutive macro-voids. The question regarding the cause of the macroscopic 6-voids, which remained unanswered in the Triatrad paper, is thus resolved. Smaller internal gaps within individual depth layers are not the subject of this work.